Generalised Sequences and Compactness Notions in Point-free Topology

by

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Declaration

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With regard to Chapter 6, the nature and scope of my contribution were as follows:

Nature of contribution	Extent of contribution
	33.3 %

The following co-authors have contributed to chapter 6:

Name	e-mail address	Nature of contribution	Extent of contribution
Mark Sioen			33.3%
David Holgate			33.3%

The undersigned hereby confirm that

1. the declaration above accurately reflects the nature and extent of the contributions of the candidate and the co-authors to chapter 6,
2. no other authors contributed to chapter 6 besides those specified above, and
3. potential conflicts of interest have been revealed to all interested parties and that the necessary arrangements have been made to use the material in chapter 6 of this dissertation.

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Abstract

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While sequences and naturally associated notions like convergence and clustering have received extensive attention in classical topology, the same cannot be said for the point-free setting. The aim of this dissertation is to introduce sequences and related sequential notions in frames and to establish the extent to which point-free sequences can characterise countable compactness notions.

Furthermore, we will introduce the point-free Dini Property and Strong Dini Property and employ these properties to characterise weaker compactness notions. We also characterise those completely regular frames satisfying the Stone-Weierstrass property.

Opsomming

Veralgemeende Rye en Kompaktheidskonsepte in Puntvrye Topologie

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Terwyl rye en natuurlike geassosieerde konsepte soos konvergensie en groepering ("*clustering*") omvattende aandag in klassieke topologie geniet het, kan dieselfde nie vir die puntvrye omgewing gesê word nie. Die doel van hierdie proefskrif is om veralgemeende rye en vewante konsepte in te voer en vas te stel tot watter mate hierdie puntvrye rye aftelbare vorme van kompaktheid kan karakteriseer.

Daarbenewens sal ons die puntvrye Dini Eienskap ("*Dini Property*") en Sterk Dini Eienskap ("*Strong Dini Property*") ondersoek en swakker vorme van kompaktheid daarmee karakteriseer. Ons karakteriseer ook die volledig reguliere puntvrye ruimtes met die Stone-Weierstrass Eienskap ("*Stone-Weierstrass Property*").

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"The woods are lovely, dark, and deep.
But I have promises to keep,
And miles to go before I sleep,
And miles to go before I sleep."

Robert Frost

Contents

Introduction and Synopsis

Compactness and its various equivalent characterisations as we know it today is due to the contributions of various mathematicians dating back to the late 19th and early 20th century. In Analysis, and related fields, its early importance lies in the fact that it can transfer local information about a function (i.e. continuity) to global information about the function (i.e. uniform continuity) [66]. For mathematicians studying more generalised spaces, its significance lies in the effect it has on the cardinality of covers.

Broadly speaking, the development of compactness, according to [66], is the result of research foci that fall within one of the following two camps: (i) a functional approach which concerns research on continuous functions defined on intervals (closed and bounded) of the real line, and (ii) a covering approach, being the study of topological aspects of the real line, such as covers of open intervals.

The French mathematician Maurice Fréchet was first to define (countable) compactness in [34] as a generalisation of a theorem (now known as the Bolzano-Weierstrass Property) to abstract topological spaces. This theorem can be dated back to a lemma (now known as the Greatest Lower Bound Property) that Bernard Bolzano [67] used in his proof of the Intermediate Value Theorem in 1817 and which was rediscovered 50 years later by Karl Weierstrass [75].

While Bolzano and Weierstrass explored sequences on the real line, other mathematicians considered covers of it. In 1872, Eduard Heine proved that a continuous function defined on a closed interval is uniformly continuous. Peter Gustav Lejeune Dirichlet [30], however, was first to prove this result in his 1852 lectures, but his proof was only published in 1904. This is the most likely reason why the result is not credited to Dirichlet. Their proofs required the result that if a countable family of intervals covers a closed and bounded interval, then a finite subfamily covers it. The significance of this result was realised by Émile Borel [21] in 1894. Pierre Cousin [30] and Henri Lebesgue [42] saw the possibility of generalising this result to arbitrary covers and provided a proof of it in 1895 and 1902, respectively. In modern terminology, the result states that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded (now known as the Heine-Borel Theorem).

Interpretations of compactness in spaces of continuous functions were made by Cesare Arzelà and Giulio Ascoli in the late 19th century. While it is unclear whether Arzelà and Ascoli knew how their work was connected with compactness, according to [66], it is clear that they were influenced by Fréchet. Their investigations lead to a generalisation of the Bolzano-Weierstrass Theorem, now known as the Arzelà-Ascoli Theorem, to families of continuous functions. It states that every bounded and equicontinuous sequence of functions in $C^0[a,b]$ has a uniformly convergent subsequence. As a consequence of this theorem, we have the following characterisation: A subset of $C^0[a,b]$ is compact if and only if it is closed, bounded and equicontinuous. The sufficiency of this theorem was proved by Ascoli [4] in 1884 and necessity by Arzelà [3] in 1889. Hilbert apparently independently discovered this property in 1900 [66].

In the early 20th century Fréchet [66] provided definitions of what is now known as countable and sequential compactness while he was exploring generalised versions of compactness in metric spaces. On the other hand, his Russian contemporaries Pavel Alexandroff and Pavel Urysohn were investigating compactness in arbitrary topological spaces. It is during this time that the important work by Felix Hausdorff [37] on topological and metric spaces was published.

In 1923, Alexandroff and Urysohn [2] provided the open-cover definition of compactness for topological spaces, i.e. every open cover has a finite subcover, and showed that under appropriate conditions Fréchet's notion of compactness followed from this definition. Not only was this version of compactness stronger, but it could be stated requiring very little technical information, and has thus become the favoured definition among mathematicians.

The importance of compactness and other related notions is evident considering the vast number of papers published on the subject in various branches of mathematics. Research into compactness notions in the point-free setting followed from this keen interest. As the point-free setting constitutes the framework for this dissertation, the author finds it appropriate to include a brief account on the early development of this setting.

The origin of point-free topology can be traced to the work of Felix Hausdorff [37] who was first to consider the notion of (open) set (open neighbourhood) as primitive in 1914 [49]. The research of Marshall Stone on the connection between algebra (lattice theory) and topology contributed to the early development of point-free topology. During the 1930's Stone published papers on the topological representation of Boolean algebras [72, 73] and distributive lattices [74]. Henry Wallman [76] applied lattice theory to define a compactification of a T_1-topological space in his 1938 publication.

The research contributions during the 1940's by Karl Menger [55], Chen McKinsey and Alfred Tarski [54] further narrowed the gap between topology and algebra. Moreover, Georg Nöbeling [56] wrote the first textbook on topology from a lattice-theoretic perspective.

It was only in the late 1950's that a fundamental shift came about in the research approach to topology. This shift gave rise to what is now known as *Frame theory* (*Locale theory*). The publications by Charles Ehresmann [32] and his student, Jean Bénabou [19], contain some of the earlier work on frame (locale) theory. The frame (locale) theoretic setting has the advantage that proofs are generally more constructive and do not require choice principles [49]. In addition, John Isbell [48] showed in 1972 that products behave better in this setting than in topology.

From the 1970's onward, interest grew rapidly and great emphasis was placed on enriching the point-free setting by providing a frame (locale) counterpart to classical notions. Hugh Dowker and Dona Strauss published research on separation axioms [25], sums and products [26] for frames. Bernard Banaschewski and Christopher Mulvey [14] published on the Stone-Čech compactification of locales and Peter Johnstone's *Stone spaces* [50] is still a primary source of reference to point-free topology today, as well as the more recent book by Picado and Pultr [59]. In more recent years weaker compactness notions have received considerable attention in frames. Uniform, nearness, metric and σ-frames also enjoyed interest amongst authors and continue doing so. We refer the reader to Picado and Pultr [59] for more detail.

We conclude the introduction with a synopsis of this dissertation:

As the title of this dissertation suggests, our aim is to introduce sequences and related sequential notions in the point-free setting. To the best of our knowledge, this is an endeavour that has not been the focus of research of any other candidate. Moreover, we aim to explore the extent to which important (weaker) compactness notions, and other tools which we develop, can be characterised by point-free sequences.

Chapter one is dedicated to providing the essential background material on the necessary subject matter in this dissertation. The reader will find introductory content on the structure of frames, frame homomorphisms and point-free separation notions. Sublocales, thought of as generalised subspaces, will often be mentioned throughout this dissertation and therefore a brief account of various equivalent notions is also provided. Frame coproducts are essential to the well-known Kuratowski-Mrówka characterisation of compactness and thus a thorough account of the construction of frame coproducts is presented. A natural and useful representation of the real numbers in the point-free setting and other related content can be found towards the end of the chapter.

In the second chapter the reader will be introduced to point-free sequences and related sequential notions with supporting motivations throughout. The chapter concludes with some basic results concerning convergence and clustering.

In chapter three our attention turns to weaker compactness notions and to determining the extent to which point-free sequences and convergence/clustering can characterise these notions.

Our focus in chapter four is directed to the inter-relatedness between extension-closed and nearly closed sublocales. This lays the groundwork for introducing almost closed sublocales which we then employ to provide a Kuratowski-Mrówka-type characterisation of compact Boolean frames.

In the fifth chapter we apply the generalised notions of points, sequences and filters in different contexts to produce a miscellany of unrelated results.

The final chapter is devoted to introducing the Dini Property and Strong Dini Property in the point-free setting. These properties are then employed to characterise weaker compactness notions. We conclude the chapter with a characterisation of the Stone-Weierstrass property for completely regular frames. This chapter resulted in a published paper [44] by M. Sioen, D. Holgate and myself. A second paper [45], based on the material of chapters 2,3,4 and 5 is in preparation.

The content in this dissertation has been structured according to a tree-like design to aid effortless reading and to assist understanding. All formal statements, i.e. propositions, lemmas, definitions, etc. are numbered in order of their appearance in each section. For example, Proposition $a.b.c$ is the c'th item of section b in chapter a. This dissertation contains the results and contributions of various authors and, where applicable, propositions (resp. lemmas, theorems etc.) are credited accordingly. Unless otherwise indicated, where propositions (resp. lemma, theorems etc.) with proofs are credited, it is to be understood that these proofs are sourced from the credited author(s).

Chapter 1

Preliminaries

In this chapter the reader will find the essential background material, with some prior knowledge assumed, on the content in succeeding chapters. In addition, the aim is to agree on terminology and fix notation. Our notation will narrowly follow that of Pultr [62]. As a blanket reference for all frame related matters, we suggest the standard texts of Picado [59], Pultr [59, 62] and Johnstone [50]. For the convenience of the the reader, we present them with some proofs from the aforementioned texts.

1.1 Separation axioms

Let $\mathcal{U}_x$ denote the collection of neighbourhoods of $x \in X$, then a topological space (X, τ) is a

(i) T_0-*space* if and only if for all $x \neq y$ in X there exists $U \in \mathcal{U}_x$ or $V \in \mathcal{U}_y$ such that $x \notin V$ or $y \notin U$.

(ii) T_1-*space* if and only if for all $x \neq y$ in X there exists $U \in \mathcal{U}_x$ and $V \in \mathcal{U}_y$ such that $x \notin V$ and $y \notin U$.

(iii) T_2-*space* (or *Hausdorff*) if and only if for all $x \neq y$ in X there exists $U \in \mathcal{U}_x$ and $V \in \mathcal{U}_y$ such that $U \cap V = \emptyset$.

(iv) T_3-*space* if and only if X is T_0 and for all $x \in X$ and for all closed $A \subseteq X$ such that $x \notin A$ there exists $U \in \mathcal{U}_x$ and open $V \supseteq A$ such that $U \cap V = \emptyset$.

A space for which every join-irreducible closed set A in X is the closure of a unique $x \in X$, is said to be *sober*. A T_3-space with axiom T_0 removed will be called *regular*. A *completely regular space* X is one in which for any $x \in X$ and any closed set F in X, $x \notin F$, there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$.

1.2 Galois adjunction

Let A and B be partially ordered sets (posets). Monotone increasing maps $f : A \longrightarrow B$ and $g : B \longrightarrow A$ are said to be *(Galois) adjoint* if

$$f(x) \leq y \text{ if and only if } x \leq g(y).$$

This condition is easily seen to be equivalent to

$$fg(y) \leq y \text{ and } gf(x) \geq x \text{ for all } x \in A, y \in B.$$

We also say f *is the left adjoint of* g and g *is the right adjoint of* f. We recall some useful facts:

(i) g is uniquely determined by f and vice versa, justifying the use of “the” in “the left/right adjoint”,

(ii) f (resp. g) preserves all existing suprema (resp. infima) and if A, B are complete lattices, then each map that preserves all suprema (resp. infima) is a left adjoint (resp. right adjoint).

1.3 Heyting algebra and Heyting operation

Let I denote an index set and A a complete lattice. The supremum (join) of a subset $\{a_i \,|\, i \in I\} \subseteq A$ will be denoted by $\bigvee_{i\in I} a_i$ and infimum (meet) by $\bigwedge_{i\in I} a_i$. We write $a_1 \vee a_2 \vee \ldots \vee a_j$ (resp. $a_1 \wedge a_2 \wedge \ldots \wedge a_j$) to denote the join (resp. meet) for finite $\{a_i \,|\, i \in I\}$.

A *Heyting algebra* is a non-empty lattice A with an additional binary relation (referred to as the *Heyting operation*) $a \to b$ satisfying

$$a \wedge b \leq c \text{ if and only if } a \leq b \to c.$$

Not all lattices admit a Heyting operation, but if A is complete, then the infinite distributive law, given by

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i), \qquad (*)$$

is a necessary and sufficient condition for its existence.

1.4 Frames and frame homomorphisms

A *frame* L is a complete lattice that satisfies the infinite distributive law $(*)$. A frame necessarily has a *top* and *bottom* element, which we denote by 1 and 0, respectively. A classical example of a frame is the open set lattice that is brought about when any topology on a set is ordered by inclusion.

The homomorphisms that preserve the frame structure will be referred to as *frame homomorphisms* or *frame maps*. To be precise, if L and M are frames, then a map $h : L \longrightarrow M$ is said to be a *frame homomorphism* if it preserves finite meets (including 1) and arbitrary joins (including 0). That is, for $a, b \in L$ and $\{c_i \,|\, i \in I\} \subseteq L$,

$$h(a \wedge b) = h(a) \wedge h(b),\ h(1) = 1 \ \text{ and } \ h(\bigvee_{i \in I} c_i) = \bigvee_{i \in I} h(c_i).$$

A frame map $h : L \longrightarrow M$ is said to be *dense* if $h(a) = 0$ implies that $a = 0$. The category of frames and frame homomorphisms will be denoted by **Frm** and the category of topological spaces and continuous maps by **Top**.

Since a frame homomorphism $h : L \longrightarrow M$ preserves all joins, it has a corresponding *right adjoint* $h_* : M \longrightarrow L$ defined by

$$h_*(b) = \bigvee \{a \in L \,|\, h(a) \leq b\}.$$

This map is not necessarily a frame homomorphism, but it does preserve arbitrary meets, as mentioned earlier. For $a \in L$, define the set $\uparrow a \subseteq L$ by

$$\uparrow a = \{b \in L \mid b \geq a\}.$$

It has a as the bottom element, 1_L as the top element and is closed under all finite meets and arbitrary joins, making it a frame.

1.5 Functors Ω : Top $\longrightarrow$ Loc and Σ : Loc $\longrightarrow$ Top

There exists a contravariant functor Ω: **Top** $\longrightarrow$ **Frm** such that a topological space is sent to its frame of open sets and a continuous map is sent to the pre-image map:

$$\begin{array}{ccc} \mathbf{Top} & \xrightarrow{\Omega} & \mathbf{Frm} \\ X & \mapsto & \Omega(X) \\ f : X \longrightarrow Y & \mapsto & \Omega(f) : \Omega(Y) \longrightarrow \Omega(X) \end{array}$$

where $\Omega(f)(U) = f^{-1}(U)$.

The category **Loc** of *locales* and *localic maps* is the dual of **Frm**. This gives rise to a covariant functor Ω : **Top** $\longrightarrow$ **Loc**. The two categories, **Loc** and **Frm**, have the same objects, but arrows are "turned around" in **Loc**, with far-reaching consequences. Although this act is mathematically trivial, to think backwards is not always clear and straightforward. Picado and Pultr [58] comments that morphisms in **Loc** are frame homomorphisms taken backwards, which may obscure the intuition. However, in [58], they also demonstrate that a covariant approach can be of great benefit. There is no preferential category amongst authors; depending on their need, some authors operate in both categories.

If we restrict Ω to **Sob**, the category of sober spaces and continuous maps, then

$$\Omega : \mathbf{Sob} \longrightarrow \mathbf{Loc}$$

is a full embedding, see Prop. 3.1.2. The spectrum functor

$$\begin{array}{ccc} \mathbf{Frm} & \xrightarrow{\Sigma} & \mathbf{Top} \\ L & \mapsto & (\Sigma L, \{\Sigma_a \,|\, a \in L\}) \\ h : L \longrightarrow M & \mapsto & \Sigma h : \Sigma M \longrightarrow \Sigma L \end{array}$$

where ΣL denotes the set of all maps $\alpha : L \longrightarrow \mathbf{2}$, $\Sigma_a = \{\alpha : L \longrightarrow \mathbf{2} \,|\, \alpha(a) = 1\}$ and $(\Sigma h)(\alpha) = \alpha h$, allows for full reconstruction of sober spaces and continuous maps. We refer to [62] for full details. Moreover, we have

Theorem 1.5.1. $\Sigma : \mathbf{Loc} \longrightarrow \mathbf{Top}$ *is a right adjoint to* $\Omega : \mathbf{Top} \longrightarrow \mathbf{Loc}$.

1.6 Pseudocomplements

In chapters to come, the reader will often come across the *pseudocomplement* a^* of an element $a \in L$. We define the pseudocomplement by

$$a^* = \bigvee\{x \in L \,|\, x \wedge a = 0\}.$$

Consequently, we have $a \wedge a^* = 0$, but it is not necessarily the case that $a \vee a^* = 1$. An element of a frame has precisely one pseudocomplement. In the case where $a \vee a^* = 1$, then we say the element a is *complemented*. For every frame L we have the following properties: For any $a \in L$,

(i) $a \leq a^{**}$,

(ii) $a \leq b \Longrightarrow a^* \geq b^*$,

(iii) $a^{***} = a^*$,

(iv) $0^* = 1$ and $1^* = 0$.

Another property of the pseudocomplement of a frame that we will employ freely is

$$(\bigvee_{i \in I} a_i)^* = \bigwedge_{i \in I} a_i^*.$$

For an element $a \in L$, if $a^{**} = a$, then we say a is *regular*. If every element of a frame is regular, then we say the frame is *Boolean*.

1.7 Booleanization

For any a, b in frame L, we have

$$(a \wedge b)^{**} = a^{**} \wedge b^{**}.$$

A Boolean frame $\mathcal{B}L$ can be constructed from any frame L by defining

$$\mathcal{B}L = \{a \in L \,|\, a = a^{**}\}.$$

This frame has finite meets which coincide with finite meets in L, as can be seen from the formula above. For any $\{a_i \,|\, i \in I\} \subseteq \mathcal{B}L$, the supremum in $\mathcal{B}L$ is defined by

$$\overline{\bigvee_{i\in I}} a_i = (\bigvee_{i\in I} a_i)^{**}.$$

The (surjective) frame homomorphism $\beta_L : L \longrightarrow \mathcal{B}L$, where $\beta_L(a) = a^{**}$, is often referred to as the *Booleanization of* L.

1.8 Nuclei, sublocale maps and sublocale sets

Subobjects in a category are often represented by extremal monomorphisms, the dual being extremal epimorphisms. In **Top** the extremal monomorphisms (which are the initial one-one maps) represent the topological subspaces.

Lemma 1.8.1. *For a frame homomorphism h the following statements are equivalent:*

(i) h is onto,

(ii) h is an extremal epimorphism

Combining this result from [58], with the fact that the embedding $j : A \subseteq X$ is naturally associated with an onto frame homomorphism $\Omega(j)$, where

$$\Omega(j) = (U \mapsto U \cap A) : \Omega(X) \longrightarrow \Omega(A),$$

naturally leads to defining a *sublocale (map)* (or *quotient (map)*) of a frame L as a surjective frame homomorphism $h : L \longrightarrow M$. For any $a \in L$, a *closed* sublocale, the point-free analogue of a closed subspace, is represented by

$$\check{a} = (x \mapsto x \vee a) : L \longrightarrow \uparrow a.$$

The monomorphisms in **Frm** are exactly given by the one-one frame homomorphisms and one can show that **Frm** has an (extremal epi, mono)-factorisation structure (system), see [1] for more on factorisation structures. Hence, any frame homomorphism $h : L \longrightarrow M$ factorises as

$$\begin{tikzcd} L \arrow[rr, "h"] \arrow[dr, "\overline{h}=(x\mapsto h(x))"'] & & M \\ & h(L) \arrow[ur, hook, "k=\subseteq"'] & \end{tikzcd}$$

with $\overline{h}$ onto and k one-one, a fact that will be used in subsequent chapters.

There are at least four equivalent ways to represent sublocales, some of which can already be found in [50]. Two more representations are provided below, both of which are employed freely throughout this dissertation:

A *nucleus* on a frame L is a map $\nu : L \longrightarrow L$ such that for every $a \in L$,

(i) $a \leq \nu(a)$,

(ii) $\nu(\nu(a)) = \nu(a)$, and

(iii) $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$.

The Booleanization $\beta_L : L \longrightarrow \mathcal{B}L$ is an example of a nucleus. The following properties of the nucleus will be useful:

Lemma 1.8.2. *Given a nucleus $\nu : L \longrightarrow L$ on a frame L:*

(i) For $\{a_i \,|\, i \in I\} \subseteq L$, we have $\nu(\bigwedge a_i) = \bigwedge \nu(a_i)$.

(ii) For every $a, b \in L$, $\nu(a \to \nu(b)) = a \to \nu(b)$.

Proposition 1.8.3. *The subset $\nu(L) \subseteq L$ is a frame with infima coinciding with those of L and the suprema given by $\bigvee' a_i = \nu(\bigvee a_i)$; the restriction $\nu : L \longrightarrow \nu(L)$ is a frame homomorphism.*

A subset S of a frame L is said to be a *sublocale set* if

(i) for each $A \subseteq S$, $\bigwedge A \in S$, and

(ii) for each $a \in L$ and $b \in S$, $a \to b \in S$.

For a sublocale map $h : L \longrightarrow M$ and sublocale set $S \subseteq L$, the translation between the two is given by:

$$h \mapsto h_*(M), \text{ where } h_* \text{ is the corresponding right adjoint, and}$$
$$S \mapsto j_S^* : L \longrightarrow S, \text{ the left adjoint of } j : S \hookrightarrow L.$$

Another useful translation is that between a nucleus $\nu : L \longrightarrow L$ and sublocale map $h : L \longrightarrow M$:

$$\nu \mapsto h_\nu = \nu \text{ restricted to } L \longrightarrow \nu(L), \text{ and}$$
$$h \mapsto \nu_h = (x \mapsto h_* h(x)) : L \longrightarrow L$$

We refer the reader to [58] for an enlightening and complete survey on the topic.

1.9 Ideals and filters

An *ideal* I of a frame L is a subset of L satisfying:

(i) $0 \in I$.

(ii) If $a, b \in I$, then $a \vee b \in I$.

(iii) If $a \in I$ and $b \in L$ with $b \leq a$, then $b \in I$. (that is, I is a downset.)

If $1 \notin I$, then an ideal I is said to be *proper*. Denote by $\mathfrak{J}L$ the set of all ideals of a frame L ordered by inclusion. A *cover* of a frame L is a subset of L whose join is the top element. If every cover of L has a finite subcover, then we say L is *compact*.

Proposition 1.9.1. *$\mathfrak{J}L$ is a compact frame.*

Proof. Trivially, the intersection of ideals is an ideal. We define the supremum by the formula

$$\bigvee_{i \in I} J_i = \left\{ \bigvee X \mid X \text{ finite}, X \subseteq \bigcup_{i \in I} J_i \right\}.$$

This set is an ideal (it is a downset since finite meet distributes over (finite) join) and any ideal J with $J_i \subseteq J$, for all i, will necessarily have $\bigvee X \in J$ for all finite

$X \subseteq \bigcup_{i\in I} J_i$. Now we show that $(\bigvee J_i) \cap K = \bigvee (J_i \cap K)$ for arbitrary ideals J_i and K: If $x = x_1 \vee \ldots \vee x_k \in (\bigvee J_i) \cap K$, with $i_1, \ldots, i_k \in I$ and $x_j \in J_{i_j}$ $(1 \leq j \leq k)$, then since K is an ideal $x_j \in J_{i_j} \cap K$ and hence $x \in \bigvee (J_i \cap K)$. The other inclusion is trivial.

Finally, let $\{J_i \,|\, i \in I\}$ be a cover of $\mathfrak{J}L$. Then, $1 \in L = \bigvee J_i$ and there are $i_1, \ldots, i_k \in I$ and $x_j \in J_{i_j}$ $(1 \leq j \leq k)$ such that $1 = x_1 \vee \ldots \vee x_k$. Consequently, $1 \in \bigvee_{j=1}^{k} J_{i_j}$ and by definition of an ideal $L = \bigvee_{j=1}^{k} J_{i_j}$.

□

Define a frame homomorphism $v_L : \mathfrak{J}L \longrightarrow L$ by setting $v_L(I) = \bigvee I$. Then we have

Lemma 1.9.2. *v_L is a dense sublocale map (that is, a dense onto frame homomorphism).*

Proof. Define a monotone map $\beta : L \longrightarrow \mathfrak{J}L$ by setting $\beta(a) = \downarrow a$. It is easy to see that $v_L \beta(a) = a$ for all $a \in L$, and $\beta v_L(J) \supseteq J$ for all J. Consequently, v_L is onto and is the left Galois adjoint of β and hence preserves all suprema.

We have $v_L(L) = 1$ and for all $J_1, J_2 \in \mathfrak{J}L$, by definition of an ideal,

$$v_L(J_1) \wedge v_L(J_2) = \bigvee \{a \wedge b \,|\, a \in J_1, b \in J_2\} \leq \bigvee \{c \,|\, c \in J_1 \cap J_2\} = v_L(J_1 \cap J_2).$$

Trivially, we have $v_L(J_1 \cap J_2) \leq v_L(J_1) \wedge v_L(J_2)$ and hence v_L preserves finite meets. Finally, if $v_L(J) = 0$, then necessarily $J = \{0\}$, the bottom of $\mathfrak{J}L$.

□

The dual notion of an ideal is called a *filter*, that is, a subset $F \subseteq L$ that satisfies:

(i) $1 \in F$.

(ii) If $a, b \in F$, then $a \wedge b \in F$.

(iii) If $a \in F$ and $b \in L$ with $a \leq b$, then $b \in F$. (That is, F is an upset.)

We say a filter $F \subseteq L$ is *proper* if $0 \notin F$ and *completely prime* if $\bigvee_{i \in I} a_i \in F$ implies that $a_i \in F$, for some i. If the previous condition holds for finite joins only, then we say F is *prime*. The (completely) prime property will also extend to upsets in coming chapters. Throughout this dissertation, all filters are to be assumed proper.

1.10 Regularity and complete regularity

For a frame L and elements $a, b \in L$, we write $a \prec b$ if $a^* \vee b = 1$, or equivalently, $a \wedge x = 0$ and $b \vee x = 1$ for some $x \in L$. Next we turn to some properties of the (*rather below*) relation $\prec$ that will prove to be useful in chapters to come:

(i) $a \prec b \Rightarrow a \leq b$, and for any a, $0 \prec a \prec 1$.

(ii) $x \leq a \prec b \leq y \Rightarrow x \prec y$.

(iii) If $a \prec b$, then $b^* \prec a^*$.

(iv) If $a \prec b$, then $a^{**} \prec b$.

(v) If $a_i \prec b_i$ for $i = 1, 2$, then $a_1 \vee a_2 \prec b_1 \vee b_2$ and $a_1 \wedge a_2 \prec b_1 \wedge b_2$.

A frame L is said to be *regular* if for each $a \in L$,

$$a = \bigvee \{b \in L \mid b \prec a\}.$$

For a topological space X and $U, V \in \Omega(X)$, the pseudocomplement in $\Omega(X)$ is given by $U^* = X \backslash \overline{U}$ and hence $V \prec U$ if and only if $\overline{V} \subseteq U$. Point-free regularity is therefore an extension of classical regularity.

Another separation property, that is an extension of the classical one, is *complete regularity*. If a, b are elements of a frame L, then we write $a \prec\!\prec b$ and say that "a is *completely below* b" if there are $a_r \in L$ for each dyadic rational r in the interval $[0, 1]$ such that

$$a_0 = a, a_1 = b \text{ and } a_r \prec a_s \text{ for } r < s.$$

We then say L is *completely regular* if for any $a \in L$, we have

$$a = \bigvee\{b \in L \mid b \prec\prec a\}.$$

The properties that hold true for the relation $\prec$, also hold true for $\prec\prec$:

(i) $a \prec\prec b \Rightarrow a \leq b$, and for any a, $0 \prec\prec a \prec\prec 1$.

(ii) $x \leq a \prec\prec b \leq y \Rightarrow x \prec\prec y$.

(iii) If $a \prec\prec b$, then $b^* \prec\prec a^*$.

(iv) If $a \prec\prec b$, then $a^{**} \prec\prec b$.

(v) If $a_i \prec\prec b_i$ for $i = 1, 2$, then $a_1 \vee a_2 \prec\prec b_1 \vee b_2$ and $a_1 \wedge a_2 \prec\prec b_1 \wedge b_2$.

1.11 Coproducts in frames

To construct coproducts in **Frm** we start out with the coproducts in $\mathbf{SLat}_1$, the category of meet-semilattices with a top element and $(1, \wedge)$-preserving maps. We then extend this coproduct to take into account the join requirements of the frame and frame homomorphisms.

Coproducts in $\mathbf{SLat}_1$ are given by the following: Let L_i, $i \in J$, be meet-semilattices. Set

$$\textstyle\prod'_{i \in J} L_i = \{(a_i)_{i \in J} \in \prod_{i \in J} L_i \mid a_i = 1 \text{ for all but finitely many } i\}$$

and define, for a fixed $j \in J$ and any $a \in L_j$,

$$\gamma_j : L_j \longrightarrow \textstyle\prod'_{i \in J} L_i \text{ by setting } (\gamma_j(a))_i = \begin{cases} a & \text{for } i = j, \\ 1 & \text{otherwise.} \end{cases}$$

If $h_j : L_j \longrightarrow M$ are morphisms in $\mathbf{SLat}_1$, then it is easy to verify that

$$h' : \textstyle\prod'_{i \in J} L_i \longrightarrow M \tag{$*$}$$

$$(a_i)_{i \in J} \mapsto \bigvee_{i \in J} h_i(a_i)$$

is the unique $(1, \wedge)$-homomorphism such that $h'\gamma_j = h_j$.

Let S be a meet-semilattice with top element. Then we denote the *frame of downsets* in S by

$$\mathfrak{D}(S) = (\{A \subseteq S \,|\, A = \downarrow A\}, \subseteq)$$

and define a $(1, \wedge)$-homomorphism $\eta : S \longrightarrow \mathfrak{D}(S)$ by setting $\eta(a) = \downarrow a$.

Proposition 1.11.1. *For every (bounded) meet-semilattice S, for every frame L and for every $(1, \wedge)$-homomorphism $h : S \longrightarrow L$ there is exactly one frame homomorphism $g : \mathfrak{D}(S) \longrightarrow L$ such that $g(\downarrow a) = h(a)$, namely the mapping given by $g(A) = \bigvee\{h(a) \,|\, a \in A\}$ for all $A \in \mathfrak{D}(S)$.*

As a final step to constructing coproducts in **Frm**, we need to identify prescribed couples of elements. This can be done as follows:

If L is a frame and $R \subseteq L \times L$ a relation, then an element $s \in L$ is said to be *saturated* (more accurately, *R-saturated*) if

$$\forall a, b, c \in L \quad aRb \Rightarrow (a \wedge c \leq s \text{ if and only if } b \wedge c \leq s).$$

Some authors also refer to the element $s \in L$ as *coherent* (or *R-coherent*).

Proposition 1.11.2. *$\nu_R : L \longrightarrow L$, defined by*

$$\nu_R(a) = \bigwedge\{s \text{ saturated } \,|\, a \leq s\},$$

is a nucleus.

Now set

$$L/R = \nu_R(L) = \{a \in L \,|\, \nu_R(a) = a\}.$$

Theorem 1.11.3. *L/R is a frame and the restriction of ν_R to $L \longrightarrow L/R$ is a sublocale map. If aRb then $\nu_R(a) = \nu_R(b)$ and for every frame homomorphism $h : L \to M$ such that $aRb \Rightarrow h(a) = h(b)$, there is a (unique) frame homomorphism $\overline{h} : L/R \longrightarrow M$ such that $\overline{h}\nu_R = h$. Moreover, $\overline{h}(a) = h(a)$ for all $a \in L/R$.*

Proof. The first statement follows from Prop. 1.8.3 and Prop. 1.11.2. Further, if aRb then $b \leq \nu_R(a)$ since $a \leq \nu_R(a)$ and $\nu_R(a)$ is saturated. Hence $\nu_R(b) \leq \nu_R(a)$ and by symmetry $\nu_R(b) = \nu_R(a)$.

Let $h : L \longrightarrow M$ be such that $aRb \Rightarrow h(a) = h(b)$. Set

$$\sigma(x) = \bigvee \{y \in L \,|\, h(y) \leq h(x)\}.$$

Obviously

$$x \leq \sigma(x) \text{ and } h\sigma(x) = h(x). \tag{$**$}$$

Let aRb and $a \wedge c \leq \sigma(x)$. Then $h(b \wedge c) = h(a \wedge c) \leq h\sigma(x) = h(x)$ and hence $b \wedge c \leq \sigma(x)$. Thus, $\sigma(x)$ is saturated. Combining this fact with $(**)$ we obtain that $x \leq \nu_R(x) \leq \sigma(x)$ and hence

$$h(x) \leq h\nu_R(x) \leq h\sigma(x) = h(x).$$

□

Now, on the downset frame $\mathfrak{D}(\prod'_{i\in J} L_i)$ consider the binary relation

$$R = \left\{ \left(\eta\gamma_j(\bigvee_{m\in M} a_m), \bigcup_{m\in M} \eta\gamma_j(a_m) \right) \,|\, j \in J, a_m \in L_j \right\}$$

where M is any set of cardinality at most $|\, L_j \,|$ and set

$$\bigoplus_{i\in J} L_i = \mathfrak{D}(\textstyle\prod'_{i\in J} L_i)/R.$$

Let $\nu_R : \mathfrak{D}(\prod'_{i\in J} L_i) \longrightarrow \bigoplus_{i\in J} L_i$ be the nucleus homomorphism from Thm. 1.11.3.

Observation 1.11.4. The $\iota_j = \nu_R\eta\gamma_j$ are frame homomorphisms.

Proposition 1.11.5. *The system* $(\iota_j : L_j \longrightarrow \bigoplus_{i\in J} L_i)_{j\in J}$ *is a coproduct in* **Frm**.

Proof. Consider the diagram
where h_j are some frame homomorphisms, $h^{'}$ is the $\mathbf{SLat}_1$-homomorphism at $(*)$ and h'' is the frame homomorphism from Prop. 1.11.1. We have

$$h''(\bigvee_m \eta\gamma_j(a_m)) = \bigvee \{h'((b_i)_{i\in J}) \,|\, (b_i)_{i\in J} \leq \gamma_j(a_m) \text{ for some } m \in M\}$$

$$\begin{tikzcd} L \arrow[r, "\gamma_j"] \arrow[d, "h_j"'] & \prod'_{i\in J} L_i \arrow[r, "\eta"] \arrow[d, "h'"'] & \mathfrak{D}(\prod'_{i\in J} L_i) \arrow[r, "\nu_R"] \arrow[d, "h''"'] & \bigoplus_{i\in J} L_i \arrow[d, "h"'] \\ M \arrow[r, equal] & M \arrow[r, equal] & M \arrow[r, equal] & M \end{tikzcd}$$

$$= \bigvee_m h'\gamma_j(a_m) = \bigvee_m h_j(a_m) = h_j(\bigvee_m a_m) = h'\gamma_j(\bigvee_m a_m) = h''\eta\gamma_j(\bigvee_m a_m)$$

and hence by Thm. 1.11.3 there is a frame homomorphism h such that $h\circ\nu_R = h''$. Therefore,

$$h\circ\iota_j = h\nu_R\eta\gamma_j = h''\eta\gamma_j = h'\gamma_j = h_j.$$

The uniqueness follows from the fact that all the elements of $\mathfrak{D}(\prod'_{i\in J} L_i)$ are joins of finite meets of the $\eta\gamma_j(a)$. Thus all the elements of $\bigoplus_{i\in J} L_i$ are joins of finite meets of the $\iota_j(a)$, for $j\in J, a\in L_j$.

$\square$

For finite systems we write

$$L\oplus M\,,\ L_1\oplus L_2\,,\ L_1\oplus L_2\oplus L_3\,,\ \text{etc.}$$

Because of the possibility that the sets M, in the definition of the relation R, may be empty, we have $({\downarrow}\gamma_j(0),\emptyset)\in R$ for all j. Set

$$\mathbf{0} = \left\{(a_i)_{i\in J}\in \textstyle\prod'_{i\in J}L_i \mid \text{ there exists } i, a_i = 0\right\}.$$

$\mathbf{0}$ is the smallest saturated set and hence the bottom of the coproduct.

Proposition 1.11.6 (Picado [59], Pultr [59,62])**.** *For every $(a_i)_{i\in J}\in\prod'_{i\in J} L_i$ the set*

$$\oplus_{i\in J}a_i = {\downarrow}(a_i)_{i\in J}\cup\mathbf{0}$$

is saturated.

If the index set J is finite, we write

$$a\oplus b\,,\ a_1\oplus a_2\,,\ a_1\oplus a_2\oplus a_3\,,\ \text{etc.}$$

We know exactly how the saturated sets look like: $U\in\mathfrak{D}(L_1\times L_2)$ is saturated if and only if

(i) $(x, 0), (0, y) \in U$ for all $x \in L_1, y \in L_2$, and

(ii) $(\bigvee_{i\in J} x_i, y) \in U$ (resp. $(x, \bigvee_{i\in J} y_i) \in U$) whenever $(x_i, y) \in U$ (resp. $(x, y_i) \in U$) for all $i \in J$.

Some very useful facts concerning the coproduct:

Proposition 1.11.7. *(i) For each $U \in \bigoplus_{i\in J} L_i$,*

$$U = \bigcup\{\oplus_{i\in J} a_i \mid \oplus_{i\in J} a_i \leq U\} = \bigvee\{\oplus_{i\in J} a_i \mid \oplus_{i\in J} a_i \leq U\}.$$

(ii) $\oplus_{i\in J} a_i = \bigwedge_{i\in J} \iota_i(a_i)$.

(iii) For binary coproducts we have $\bigvee_{i\in J}(a_i \oplus b) = (\bigvee_{i\in J} a_i) \oplus b$ and $\bigvee_{i\in J}(a \oplus b_i) = a \oplus (\bigvee_{i\in J} b_i)$; consequently

$$(\bigvee_{i\in J} a_i) \oplus (\bigvee_{i\in J} b_i) = \bigvee_{i,j\in J} (a_i \oplus b_j).$$

(iv) If $\oplus_{i\in J} a_i \neq \mathbf{0}$, then

$$\oplus_{i\in J} a_i \leq \oplus_{i\in J} b_i \Rightarrow \forall i,\ a_i \leq b_i.$$

(v) The codiagonal homomorphism $\nabla : \bigoplus_{i\in J} L_i \longrightarrow L$, with $L_i = L$ for all i, is given by the formula

$$\nabla(\oplus_{i\in J} a_i) = \bigwedge_{i\in J} a_i.$$

1.12 The frame of reals

We denote the standard ordered set of rationals and reals by $\mathbb{Q}$ and $\mathbb{R}$, respectively. The idea behind defining the point-free reals is to take $\mathbb{Q}$ and to take the set of open intervals with rational endpoints for the basic generators. The *frame of reals* is the frame $\mathcal{L}(\mathbb{R})$ generated by all ordered pairs (p, q) where $p, q \in \mathbb{Q}$, subject to the relations

(i) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(ii) $(p,q) \vee (r,s) = (p,s)$ whenever $p \leq r < q \leq s$,

(iii) $(p,q) = \bigvee\{(r,s) \,|\, p < r < s < q\}$,

(iv) $1 = \bigvee\{(p,q) \,|\, p,q \in \mathbb{Q}\}$.

By using classical logic it is mentioned in [8] and [41] that one can prove $\mathcal{L}(\mathbb{R})$ is isomorphic to $\Omega(\mathbb{R})$ ($\mathbb{R}$ equipped with the Euclidean topology). Nevertheless, the advantage of $\mathcal{L}(\mathbb{R})$ is that it is presented by using generators and relations. For a thorough study of the frame of reals, we refer the reader to [8].

For any frame L, a frame homomorphism $\alpha : \mathcal{L}(\mathbb{R}) \longrightarrow L$ will be referred to as a *continuous real function* on L and the family of all continuous real functions on L will be denoted by $\mathcal{R}L$. A continuous real function $\alpha \in \mathcal{R}L$ is said to be *bounded* if there exists $p,q \in \mathbb{Q}$ such that $\alpha(p,q) = 1$.

The reader will often come across the following notation in $\mathcal{L}(\mathbb{R})$:

$$(p,-) = \bigvee\{(p,q) \,|\, p < q \in \mathbb{Q}\},$$
$$(-,q) = \bigvee\{(p,q) \,|\, q > p \in \mathbb{Q}\}.$$

For $\alpha, \beta \in \mathcal{R}L$ and $\langle p,q\rangle = \{t \in \mathbb{Q} \mid p < t < q\}$, the operations on $\mathbb{Q}$ are used to define the following operations on $\mathcal{R}L$, making it into a *lattice*-ordered ring or ℓ-ring:

(i) For $\diamond \in \{+,\cdot,\wedge,\vee\}$ define:

$$(\alpha \diamond \beta)(p,q) = \bigvee\{\alpha(r,s) \wedge \beta(t,u) \mid \langle r,s\rangle \diamond \langle t,u\rangle \subseteq \langle p,q\rangle\},$$

where $\langle r,s\rangle \diamond \langle t,u\rangle = \{x \diamond y \mid x \in \langle r,s\rangle, y \in \langle t,u\rangle\}$.

(ii) $(-\alpha)(p,q) = \alpha(-q,-p)$.

(iii) For every $r \in \mathbb{Q}$, an operation $\mathbf{r}$ is defined by $\mathbf{r}(p,q) = 1$ whenever $p < r < q$ and $\mathbf{r}(p,q) = 0$ otherwise.

The partial order on $\mathcal{R}L$ is completely described by the following string of equivalences: for $\alpha, \beta \in \mathcal{R}L$, we have that $\alpha \leq \beta$, that is, $\alpha \vee \beta = \beta$ if and only if

$\beta(-,t) \leq \alpha(-,t)$ for all $t \in \mathbb{Q}$, if and only if $\alpha(s,-) \leq \beta(s,-)$ for all $s \in \mathbb{Q}$.

For an $\alpha \in \mathcal{R}L$, the *absolute value* of α is defined by $|\alpha| = \alpha \vee (-\alpha)$. One can show that α is bounded if and only if there exists $n \in \mathbb{N}$ such that $|\alpha| \leq \mathbf{n}$. The *positive* and *negative part* of α is defined as usual:

$$\alpha^+ = \alpha \vee \mathbf{0} \text{ and } \alpha^- = (-\alpha) \vee \mathbf{0}.$$

$\mathcal{R}L$ can then be shown to be a commutative strong archimedean f-ring with unit $\mathbf{1}$. We refer to Bigard, Keimel and Wolfenstein [20] for more information on lattice-ordered rings. The set $\mathcal{R}^*L := \{\alpha \in \mathcal{R}L \mid |\alpha| \leq \mathbf{n} \text{ for some } n \in \mathbb{N}\}$ will be referred to as the *bounded part* of $\mathcal{R}L$.

The ring $\mathcal{R}L$ carries the *uniform uniformity* with entourages $\{(\alpha,\beta) \in \mathcal{R}L \times \mathcal{R}L \mid |\alpha - \beta| \leq \frac{1}{\mathbf{n}}\}$, $(n \in \mathbb{N})$ as its natural uniformity, and the underlying *uniform topology* as its natural topology. The uniform uniformity, resp. uniform topology, on $\mathcal{R}^*L$ are defined by taking subspaces. We also recall that for a topological space X, we have that $\mathcal{R}\Omega(X) \simeq C(X)$, resp. $\mathcal{R}^*\Omega(X) \simeq C^*(X)$.

The point-free way to deal with a point-dependent cozero set is given by the *cozero map* coz: $\mathcal{R}L \longrightarrow L$ which assigns to each $\alpha \in \mathcal{R}L$ the element

$$\mathrm{coz}(\alpha) = \alpha((-,0) \vee (0,-)).$$

We refer to the $\mathrm{coz}(\alpha)$ as the *cozero elements* of L. The *cozero part* of L, $\mathrm{Coz}L = \{\mathrm{coz}(\alpha) \,|\, \alpha \in \mathcal{R}L\}$, is a sub-$\sigma$-frame of L, assuming the Axiom of Countable Choice (ACC) (see [8] for detail). The following set of calculation rules, which can also be found in [8], will prove useful in the final chapter of this dissertation:

(i) For all $\beta \in \mathcal{R}L$: $\mathrm{coz}(\beta^+) = \beta^+(0,-) = \beta(0,-)$.

(ii) For all $\gamma \in \mathcal{R}L$ and all $r,q \in \mathbb{Q}$: $(\gamma - \mathbf{r})(-,q) = (\mathbf{r} - \gamma)(-q,-) = \gamma(-,r+q)$ and $(\gamma - \mathbf{r})(q,-) = (\mathbf{r} - \gamma)(-,-q) = \gamma(r+q,-)$.

(iii) For all $\gamma \in \mathcal{R}L$, all $r \in \mathbb{Q}$ and all $m \in \mathbb{N}$: $(m\gamma)(-,r) = \gamma(-,\frac{r}{m})$.

(iv) For all $\gamma \in \mathcal{R}L$ and all $r \in \mathbb{Q}$: $\gamma(-,r) = 1 \Rightarrow \gamma \leq \mathbf{r}$.

A frame L is called *pseudocompact* if every continuous real-valued function on L is bounded, i.e. if $\mathcal{R}L = \mathcal{R}^*L$. Moreover, L is pseudocompact if and only if Coz$L$ is a compact σ-frame, that is, if every countable cover of L by cozero elements admits a finite subcover. (See Banaschewski and Gilmour [10] for details; Banaschewski [8] contains the result for completely regular L).

1.13 Axiom of Choice

The Axiom of κ-Choice (AκC) states that the Cartesian product of at most κ non-empty sets (κ is an infinite cardinal number) is non-empty. Both (AκC) and the stronger Axiom of Choice (AC) will be explicitly mentioned, where necessary, in Chapter 6. On the other hand, the weaker Axiom of Countable Choice (ACC) will be assumed without explicit reference.

Chapter 2

Generalised sequences on frames

This chapter will serve as a preliminary study of sequences and sequential notions in the point-free setting guided by supporting motivations. The aim is to define notions that retain sufficient topological information when points are no longer at our disposal and, moreover, that will successfully encompass the most fundamental sequential properties of classical topology. To that end, the reader will be introduced to basic point-free results on related sequential notions, like convergence and clustering. To date, this study has not been the subject of any research to the best knowledge of the author.

2.1 Motivation

If X is any non-empty set, then a *sequence* in X is a map from the set of natural numbers $\mathbb{N}$ to X. We use the notation $(f(n))$ or (x_n), $n \in \mathbb{N}$, to denote a sequence in X. This suggests that an obvious definition of a sequence in **Top** would be a continuous map $f : (\mathbb{N}, \mathcal{P}(\mathbb{N})) \longrightarrow (X, \mathcal{T})$, where $\mathcal{P}(\mathbb{N})$ denotes the powerset of $\mathbb{N}$. But first, we turn our attention to points:

Let P denote the one point space $\{*\}$, then there is a natural one-one correspondence between the points x in a space X, and continuous functions

$$f_x = (* \mapsto x) : P \longrightarrow X.$$

Hence, given a sequence of points $x_1, x_2, x_3, \ldots$ we have a sequence of functions $f_{x_1}, f_{x_2}, f_{x_3}, \ldots$ and the contravariant functor $\Omega : \mathbf{Top} \longrightarrow \mathbf{Frm}$ allows us to define a sequence $(f_{x_n}^{-1})$ of frame homomorphisms where

$$f_{x_n}^{-1} : \Omega(X) \longrightarrow \Omega(P) \cong \mathbf{2},$$

and $\mathbf{2}$ denotes the two element Boolean algebra. For a sober space X, Pultr [62] proves that we can retrieve a point from a frame homomorphism $p : \Omega(X) \longrightarrow \Omega(P) \cong \mathbf{2}$:

Proposition 2.1.1. *Let X be a sober space and Y a general one. Then for each frame homomorphism $h : \Omega(X) \longrightarrow \Omega(Y)$ there is exactly one continuous map $f : Y \longrightarrow X$ such that $h = \Omega(f)$.*

Proof. Let $h : \Omega(X) \longrightarrow \Omega(Y)$ be a frame homomorphism. For $y \in Y$ set

$$\mathcal{F}_y = \{U \in \Omega(X) \,|\, y \notin h(U)\} \text{ and } F_y = \bigcup \mathcal{F}_y.$$

Since h preserves arbitrary joins we have that $y \notin h(F_y)$, and hence, for $U \in \Omega(X)$,

$$y \notin h(U) \text{ if and only if } U \subseteq F_y.$$

Since $y \notin h(F_y)$, we have that $F_y \neq X$ and if $F_y = U \cap V$, then $y \notin h(U) \cap h(V)$, because h preserves finite meet. Therefore, say $y \notin h(U)$, $U \subseteq F_y$ and F_y is meet-irreducible.

By the sobriety of X, $F_y = X \backslash \overline{\{x\}}$ for a unique $x \in X$. This defines a function $f : Y \longrightarrow X$ if we choose such x for $f(y)$, and we can write

$$y \notin h(U) \text{ if and only if } U \subseteq X \backslash \overline{\{x\}} \text{ if and only if } f(y) \notin U,$$

since U is open and therefore

$$y \in h(U) \text{ if and only if } f(y) \in U, \text{ that is}, y \in f^{-1}(U).$$

Hence, $f^{-1}(U) = h(U) \in \Omega(Y)$ and thus f is continuous and $h = \Omega(f)$. One can readily show that f is unique, since X is sober and therefore a T_0−space.

□

Remark 2.1.2. There are many sober topological spaces, and those that are not sober can be replaced by their soberification. Johnstone [50] says that by replacing a space with its soberification the open-set lattice is left unchanged, and very little damage is done to its topological properties. By pretending that all spaces are sober, as a result, little harm comes to topology. Consequently, spaces and continuous maps represent locales and localic maps to a considerable degree. This then justifies the remark that locales can be regarded as "generalised spaces". Not all frames are, however, spatial. For example, a complete Boolean algebra is spatial if and only if it is atomic.

As a consequence of Prop. 2.1.1, we have a one-one correspondence between points x in a sober space X and points $p_n : \Omega(X) \longrightarrow \mathbf{2}$, $n \in \mathbb{N}$. This naturally leads to the following definition of a point of a frame:

Definition 2.1.3. A *point* of a frame L is a frame homomorphism $p : L \longrightarrow \mathbf{2}$.

Note that ΣL in 1.5 is a topological space defined on the set of these points. Furthermore, since $\mathcal{P}(\mathbb{N}) \cong \mathbf{2}^{\mathbb{N}}$ and from Prop. 2.1.1 it follows that for a sober space X a one-one correspondence between sequences $f : \mathbb{N} \longrightarrow X$ and frame homomorphisms $s : \Omega(X) \longrightarrow \mathbf{2}^{\mathbb{N}}$ exists.

Remark 2.1.4. Since $\mathbf{2}^{\mathbb{N}}$ can equivalently be viewed as the frame theoretic product of countably many copies of $\mathbf{2}$, we have projections $\pi_n : \mathbf{2}^{\mathbb{N}} \longrightarrow \mathbf{2}$ for every $n \in \mathbb{N}$, see Figure 2.1. Hence we have a sequence (s_n) of points $s_n = \pi_n \circ s$, $n \in \mathbb{N}$. Conversely, we can find a sequence $s : \Omega(X) \longrightarrow \mathbf{2}^{\mathbb{N}}$ given a sequence of points $s_n : \Omega(X) \longrightarrow \mathbf{2}$, $n \in \mathbb{N}$.

$$\begin{tikzcd}
\Omega(X) \arrow[rr, "s_n"] \arrow[dr, "s"'] & & \mathbf{2} \\
& \mathbf{2}^{\mathbb{N}} \arrow[ur, "\pi_n"'] &
\end{tikzcd}$$

Figure 2.1: Frame theoretic product $\mathbf{2}^{\mathbb{N}}$

A sequence $s : \Omega(X) \longrightarrow \mathbf{2}^{\mathbb{N}}$ is therefore nothing more than a sequence of points $p : \Omega(X) \longrightarrow \mathbf{2}$. And, consequently, too point dependent to render a useful point-free notion.

2.2 Generalisation

For a filter F on a frame L, the characteristic function of F is given by a $(0, 1, \wedge)$-homomorphism. That is, the function $h_F : L \longrightarrow \mathbf{2}$ such that $h_F(a) = 1$ if and only if $a \in F$, preserves the bottom element and finite meets (including the top). In [13] Banaschewski and Hong introduce the concept of a T-valued (or general) filter on L, for a general frame T (of "truth values") as a $(0, 1, \wedge)$-homomorphism from L into T. Moreover, general filters turn out to be just the right tool to address convergence questions in frames, which filters previously could not [13].

For the case in which the filter F on L is completely prime, the characteristic function is a point $p : L \longrightarrow \mathbf{2}$ of L. With motivation from [13], we now generalise the notion of a point:

Definition 2.2.1. A *generalised point* of a frame L is a frame homomorphism $h : L \longrightarrow M$, with frame M non-trivial. If we have countably many generalised points of L, that is,

$$s_n : L \longrightarrow M_n \text{ with } n \in \mathbb{N},$$

then we say we have a *generalised sequence* (or *sequence* for short) on L.

As is customary, we will denote such a sequence by (s_n). It is clear that there is no shortage of sequences in **Frm**. In the case that $M_n = \mathbf{2}$, for all $n \in \mathbb{N}$, then a sequence on L is exactly a sequence of points, in the classical sense.

Remarks 2.2.2. A sequence on L can also be considered from other external point of views:

(i) Since **Frm** has products, a sequence can equivalently be thought of as the unique product homomorphism $s : L \longrightarrow \prod M_n$ where $s_n = \pi_n \circ s$.

$$\begin{tikzcd} L \arrow[rr, "s_n"] \arrow[dr, "s"'] & & M_n \\ & \prod M_n \arrow[ur, "\pi_n"'] & \end{tikzcd}$$

Figure 2.2: Unique product homomorphism s

(ii) Any frame homomorphism $h : L \longrightarrow M$ can be factorised through its image, for a given sequence $s_n : L \longrightarrow M_n$ we have a sequence of sublocale maps $e_n : L \longrightarrow s_n(L)$. Though not technically an equivalent characterisation, this viewpoint still retains much of the sequential information, in particular related to convergence and clustering as described in the next section.

$$\begin{tikzcd} L \arrow[rr, "s_n"] \arrow[dr, "e_n"'] & & M_n \\ & s_n(L) \arrow[ur, hook, "m_n"'] & \end{tikzcd}$$

Figure 2.3: Image factorisation of $s_n = m_n \circ e_n$

A notion of subsequence of a sequence on L can be defined:

Definition 2.2.3. Given a sequence (s_n), then a sequence (t_n) is said to be a *subsequence* of (s_n) if there is an increasing map $k \mapsto n_k$ such that $t_k = s_{n_k}$ for all $k \in \mathbb{N}$. We will denote this subsequence by (s_{n_k}).

Two notions synonymous with sequences are those of convergence and clustering. Point-free convergence has been the subject of research by Pultr [15], Banaschewski [12, 15] and Hong [12, 47] (amongst others) dating back to 1990. Banaschewski and Pultr [15] defined convergence in terms of completely prime filters whereas Hong [47] was first to introduce convergence of filters in frames by means of covers in 1995.

2.3 Convergence and clustering

For an arbitrary topological space X one says a sequence (x_n) in X *converges* to a point x if for all $U \in \mathcal{U}_x$ there exists $n \in \mathbb{N}$ such that $x_m \in U$ for all $m \geq n$. We express this by writing $(x_n) \to x$ and call x a *limit point* of (x_n). Unlike for metric spaces (see [51] for more on metric spaces), limit points in topological spaces may not be unique.

We say that x is a *cluster point* or *accumulation point* of (x_n) if for all $U \in \mathcal{U}_x$ and for all $n \in \mathbb{N}$ there exists $m \geq n$ such that $x_m \in U$. Clearly, if a sequence (x_n) converges to a point x, then x is a cluster point of (x_n).

Finding point-free counterparts for convergence and clustering starts out with removing all references to points:

Proposition 2.3.1. *For an arbitrary sequence (x_n) in a topological space X, the following are equivalent:*

(i) There exists $x \in X$ such that $(x_n) \to x$.

(ii) For every open cover $\mathcal{C}$ of X there exists $U \in \mathcal{C}$ and $n \in \mathbb{N}$ such that $U \in \mathcal{U}_{x_m}$, for all $m \geq n$.

Proof. $(i) \Rightarrow (ii)$: Trivial.

$(ii) \Rightarrow (i)$: Assume for all x there exists $U_x \in \mathcal{U}_x$, and for all $n \in \mathbb{N}$ there exists $m \geq n$ such that $U_x \notin \mathcal{U}_{x_m}$. We may assume that U_x is open for each x so that we have the open cover $\mathcal{C} = \{ U_x \mid x \in X \}$. Then for every $U \in \mathcal{C}$ and for all $n \in \mathbb{N}$ there exists $m \geq n$ with $U \notin \mathcal{U}_{x_m}$.

□

Proposition 2.3.2. *For an arbitrary sequence (x_n) in a topological space X, the following are equivalent:*

(i) There exists $x \in X$ such that x is a cluster point of (x_n).

(ii) *For every open cover $\mathcal{C}$ of X there exists $U \in \mathcal{C}$, and for all $n \in \mathbb{N}$ there exists $m \geq n$ such that $U \in \mathcal{U}_{x_m}$.*

Proof. Similar to the proof of Prop. 2.3.1.

□

Remark 2.3.3. Given a sequence (x_n) in X, the filter base $\mathcal{F} = \{F_n \,|\, n \in \mathbb{N}\}$, where $F_n = \{x_m \,|\, m \geq n\}$, generates the filter $\mathcal{F}_{(x_n)} = \{A \subseteq X \,|\, \exists F \in \mathcal{F}$ such that $F \subseteq A\}$. It is well-known that one can equivalently describe sequence convergence and clustering by means of filter convergence and clustering of $\mathcal{F}_{(x_n)}$. Banaschewski [12] and Hong [12, 47] observed that a filter of opens is convergent if and only if it meets every open cover of X, which they then used as a point-free definition to prove a number of results. This approach, however, cannot be readily applied to sequential convergence, since the filter $\mathcal{F}_{(x_n)}$ is not necessarily a filter of opens.

Combined with the observation that a point $f_{x_m} = (* \mapsto x_m) : P \longrightarrow X$ belongs to an open set U if and only if $f_{x_m}^{-1}(U) \neq \emptyset$, we now state the following definition:

Definition 2.3.4. A sequence (s_n) on L

(a) *converges* if for any cover C of L there exists $a \in C$ and $n \in \mathbb{N}$ such that $s_m(a) \neq 0$ for all $m \geq n$.

(b) *clusters* if for any cover C of L there exists $a \in C$ and for all $n \in \mathbb{N}$ there exists $m \geq n$ with $s_m(a) \neq 0$.

Remarks 2.3.5. (i) Convergence and clustering of a sequence $s_n : L \longrightarrow M_n$ can equivalently be defined via the product homomorphism $s : L \longrightarrow \prod M_n$: A sequence (s_n) on L

(a) converges if for every cover C of L there exists $a \in C$ such that $s(a)$ has all but finitely many entries not equal to 0.

(b) clusters if for every cover C of L there exists $a \in C$ such that $s(a)$ has infinitely many entries not equal to 0.

(ii) The definition manages to capture the converging (resp. clustering) behaviour of a sequence (x_n), but as a result of removing all references to points, one is unable to distinguish between two (or more) limit points.

(iii) If a sequence of points $s_n : L \longrightarrow \mathbf{2}$ converges in L, then it has a "limit" $t : L \longrightarrow \mathbf{2}$ given by

$$t(a) = 1 \text{ if and only if } \exists\, n \in \mathbb{N}\ \forall m \geq n, s_m(a) = 1.$$

This map is not necessarily a frame homomorphism, hence there might not be a "limit" point. Thereby highlighting the point-independence of the convergence.

Examples 2.3.6. (i) Consider the frame

$$L = (\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}, \subseteq)$$

and sequence $s_n : L \longrightarrow \mathbf{2}$, $n \in \mathbb{N}$, given by

$$s_{2n-1}(A) = \begin{cases} 0 & \text{if } A = \{a\}, \emptyset \\ 1 & \text{otherwise} \end{cases} \qquad s_{2n}(A) = \begin{cases} 0 & \text{if } A = \{b\}, \emptyset \\ 1 & \text{otherwise} \end{cases}$$

Then (s_n) converges to $t : L \longrightarrow \mathbf{2}$ given by

$$t(A) = \begin{cases} 0 & \text{if } A = \{a\}, \{b\}, \emptyset \\ 1 & \text{otherwise} \end{cases}$$

which is not a point of L, since t does not preserve arbitrary join.

(ii) A frame may have no points, but a sequence on a frame can still converge: Consider the Boolean algebra of regular open sets on $\mathbb{R}$, denoted by L, and any constant sequence (s_n) on L, with all generalised points s_n the same frame homomorphism. Then (s_n) converges, but L has no points.

Another stronger notion for convergence is possible. For a sequence of points $s_n : L \longrightarrow \mathbf{2}$ these two notions coincide:

Definition 2.3.7. A sequence $s_n : L \longrightarrow M_n$ *strongly converges* if for any cover C of L there exists $a \in C$ and $n \in \mathbb{N}$ such that $s_m(a) = 1$ for all $m \geq n$.

The question whether the constant identity sequence, the sequence with all points the identity homomorphism, strongly converges on a frame is raised by the definition above. The constant sequence is necessarily convergent, but may not be strongly convergent:

Example 2.3.8. Consider the 4 element Boolean algebra with the constant identity sequence. Then clearly the sequence converges, but it is not strongly convergent.

Observation 2.3.9. Factorising any sequence $s_n = m_n \circ e_n$, where $n \in \mathbb{N}$, through its image (see Figure 2.3) we observe that

$$s_n(a) = 0 \iff e_n(a) = 0 \text{ and } s_n(a) = 1 \iff e_n(a) = 1.$$

Therefore, from the point of view of convergence and clustering, we can just as well consider sequences of sublocales as opposed to frame homomorphisms. However, in the topological setting the situation is more nuanced and will be considered in greater detail in the coming chapter.

2.4 Basic results

Any sequence $s_n : L \longrightarrow M_n$ for which $L = \mathbf{2}$, will converge. Also, if L denotes the 4 element Boolean algebra, then any sequence $p_n : L \longrightarrow T_n$ clusters. If a sequence converges, then so does every subsequence. A convergent sequence also clusters, but not necessarily vice versa:

Example 2.4.1. Let L denote the 4 element Boolean algebra with $a, b \in L$, of which neither is the top nor bottom element. Define the sequence $s_n : L \longrightarrow \mathbf{2}$ with $s_{2m-1}(a) = s_{2m}(b) = 1$ and $s_{2m-1}(b) = s_{2m}(a) = 0$, for $m \in \mathbb{N}$. It is clear that (s_n) clusters, but it does not converge.

As in the topological setting, we have the following relationship between clustering and convergent sequences:

Proposition 2.4.2. *If a sequence has a convergent subsequence, then it clusters.*

Proof. Take arbitrary sequence (s_n) with convergent subsequence (s_{n_k}). For any cover C, there exists $a \in C$ and $n \in \mathbb{N}$ such that $s_{n_k}(a) \neq 0$, for all $k \geq n$. Take arbitrary $n' \in \mathbb{N}$. If $n' \leq n_k$, choose $m = n_k$ and if $n' > n_k$, then choose $m = n_{n'}$. □

Sequence convergence and clustering in the point-free setting also has an internal characterisation. Hong [47] defined convergence and clustering for filters in the point-free setting based on an observation pertaining to open covers. It was, however, first introduced by Herrlich [38] in a more general form concerning nearness spaces. Recall the following:

Definition 2.4.3. If $\mathcal{F}$ is filter in a topological space X, then define

$$\text{sec}\mathcal{F} = \{U \subseteq X \,|\, \forall\, F \in \mathcal{F}, U \cap F \neq \emptyset\}.$$

Proposition 2.4.4. *Consider a filter $\mathcal{F}$ in a topological space X, then*

(i) $\mathcal{F}$ converges if and only if for every open cover $\mathcal{C}$ of X, $\mathcal{C} \cap \mathcal{F} \neq \emptyset$.

(ii) $\mathcal{F}$ clusters if and only if for every open cover $\mathcal{C}$ of X, $\mathcal{C} \cap \text{sec}\mathcal{F} \neq \emptyset$.

Proof. (i) $(\Rightarrow)$: Given $\mathcal{U}_x \subseteq \mathcal{F}$ for some $x \in X$ and open cover $\mathcal{C}$ of X, since $\mathcal{C}$ is a cover there is a $C \in \mathcal{C}$ with $x \in C$. Then $C \in \mathcal{U}_x \cap \mathcal{C} \subseteq \mathcal{F} \cap \mathcal{C}$ so that $\mathcal{F} \cap \mathcal{C} \neq \emptyset$.
$(\Leftarrow)$: If for all $x \in X$, $\mathcal{U}_x \nsubseteq \mathcal{F}$, then for all x there is an open $U_x \in \mathcal{U}_x$ with $U_x \notin \mathcal{F}$. Then $\mathcal{C} := \{U_x \,|\, x \in X\}$ is an open cover of X with $\mathcal{C} \cap \mathcal{F} = \emptyset$.

(ii) $(\Rightarrow)$: Let $x \in \bigcap\{\overline{F} \,|\, F \in \mathcal{F}\}$, for some $x \in X$, and $\mathcal{C}$ an open cover of X. Then there is $C \in \mathcal{C}$ with $x \in C$, and therefore $C \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.
$(\Leftarrow)$: If $\bigcap\{\overline{F} \,|\, F \in \mathcal{F}\} = \emptyset$, then for all $x \in X$ there is an open $U_x \in \mathcal{U}_x$ and $F \in \mathcal{F}$ such that $U_x \cap F = \emptyset$. Hence $U_x \notin \text{sec}\mathcal{F}$ and $\mathcal{C} := \{U_x \,|\, x \in X\}$ is an open cover of X with $\mathcal{C} \cap \text{sec}\mathcal{F} = \emptyset$.

□

We modify Hong's definition slightly by defining convergence and clustering for upsets, as it turned out requiring filters is too strong.

Definition 2.4.5. For an upset F in a frame L, define

$$\text{sec}F = \{a \in L \,|\, \forall x \in F, a \wedge x \neq 0\}.$$

Definition 2.4.6. An upset F in a frame L is said to be

(a) *convergent* if for any cover C of L, F meets C.

(b) *clustered* if for any cover C of L, $\mathrm{sec}F$ meets C.

It is clear that every completely prime upset converges and a convergent upset is clustered. Furthermore, every upset that contains a convergent upset is convergent. Consequently, an upset containing a completely prime upset is convergent.

Given a sequence $s_n : L \longrightarrow M_n$, define sets A_s, B_s and C_s by

$$A_s = \{a \in L \,|\, \exists\, n \in \mathbb{N}\, \forall\, m \geq n, s_m(a) = 1\},$$

$$B_s = \{a \in L \,|\, \exists\, n \in \mathbb{N}\, \forall\, m \geq n, s_m(a) \neq 0\} \text{ and}$$

$$C_s = \{a \in L \,|\, \forall\, n \in \mathbb{N}\, \exists\, m \geq n, s_m(a) \neq 0\}.$$

One can easily show that A_s, B_s and C_s are upsets.

Proposition 2.4.7. *Let $s_n : L \longrightarrow M_n$ be a sequence on a frame L.*

(i) The sequence (s_n) is strongly convergent (resp. convergent) if and only if A_s (resp. B_s) converges.

(ii) The sequence (s_n) clusters if and only if C_s converges.

Proof. Trivial.

□

Observation 2.4.8. One would be interested to know if clustering of B_s is related to sequential clustering: For a sequence $s_n : L \longrightarrow \mathbf{2}$, we have the following set inclusions:

$$A_s = B_s \subseteq C_s \subseteq \mathrm{sec}A_s = \mathrm{sec}B_s.$$

Proof. It is clear that $A_s = B_s$. For any $a \in B_s$ and any $n \in \mathbb{N}$ we can find a $m \geq n$ such that $s_m(a) \neq 0$. Hence $a \in C_s$.

Take $c \in C_s$ and assume $c \notin \mathrm{sec}A_s$. Then there is $a \in A_s$ such that $a \wedge c = 0$. By definition of A_s, we have $n \in \mathbb{N}$ such that $s_m(a) = 1$ for all $m \geq n$. Also, since $c \in C_s$, there is $\overline{m} \geq n$ such that $s_{\overline{m}}(c) = 1$. Therefore $1 = s_{\overline{m}}(a) \wedge s_{\overline{m}}(c) = s_{\overline{m}}(a \wedge c) = 0$, a contradiction, and hence we are done.

□

The following example will show that $\sec B_s \nsubseteq C_s$ in general, asserting that sequence clustering is best described via the convergence of C_s:

Example 2.4.9. Consider a three element chain $\{0, b, 1\}$, which we will denote by B. Define a constant sequence $s_n : B \longrightarrow \mathbf{2}$ on B given by

$$s_n(a) = \begin{cases} 0 & \text{if } a = 0, b \\ 1 & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{N}$. Then $\sec B_s = \{b, 1\} \nsubseteq C_s = \{1\}$.

Chapter 3

Sequential and compactness notions

In the previous chapter our interest centered around finding conservative (meaning that for a topological space X, the frame $\Omega(X)$ has the frame property if and only if X has the spatial property baring the same name) point-free counterparts for introductory sequential notions and properties from the topological setting. In this chapter we turn our attention to compactness and we aim to establish to what extent sequences and convergence/clustering can characterise weaker compactness notions in frames. First we consider sequential compactness, for its obvious connection to sequences, followed by countable compactness notions.

3.1 Sequential compactness

In this section we will introduce point-free counterparts for sequential notions regarding closedness and compactness and provide conservative translations of relevant topological results. We start with the following well-known result:

Lemma 3.1.1. *If a topological space X has unique sequential limits, then X is a T_1-space.*

Proof. Take arbitrary $x \neq y \in X$ and assume $y \in \overline{\{x\}}$. Then $x \in U$ for every $U \in \mathcal{U}_y$ and as a result the constant sequence $(x, x, x, \dots)$ converges to x and y.

Since sequences have unique limits, we have $x = y$, a contradiction. Thus $y \notin \overline{\{x\}}$, and therefore $y \in X \backslash \overline{\{x\}}$. With a similar argument we can show that $x \notin \overline{\{y\}}$ and hence $x \in X \backslash \overline{\{y\}}$.

$\square$

A subspace does not necessarily inherit sobriety, to that end we need to add the T_D-axiom: For each $x \in X$ there is an $U \in \mathcal{U}_x$ such that $U \backslash \{x\}$ is open.

Lemma 3.1.2 (Hoffmann [43])**.** *For a topological space X the following conditions are equivalent:*

(i) X is both sober and T_D.

(ii) Every subspace of X is sober.

Remark 3.1.3. The T_D-axiom is weaker than T_1 and stronger than T_0. Neither of the axioms of sobriety and T_D imply the other (examples can be found in [64]).

Due to its strong point dependency, finding an equivalent point-free characterisation of a sequentially closed (sub)set would seem unlikely. We previously mentioned (see Remarks 2.3.5) that with point-free convergence one is unable to distinguish between two (or more) limit points. Thus, for spaces, it most accurately imitates sequence convergence for unique limit points. But, as we know, spaces do not exhibit this behaviour in general and therefore we require additional properties:

Proposition 3.1.4. *Let X be a sober topological space with unique sequential limits. Furthermore, consider the embedding $j : M \subseteq X$ and sublocale $h = \Omega(j) : \Omega(X) \longrightarrow \Omega(M)$. Then the following statements are equivalent:*

(i) M is sequentially closed.

(ii) For any sequence $s_n : \Omega(M) \longrightarrow \mathbf{2}$, (s_n) is convergent whenever $(s_n \circ h)$ is convergent.

Proof. $(i) \Rightarrow (ii)$: Take an arbitrary sequence $s_n : \Omega(M) \longrightarrow \mathbf{2}$ and assume $(s_n \circ h)$ converges. From Lemma 3.1.2 it follows that M is sober and hence we can construct a sequence (a_n) in M. Then (a_n) is also a sequence in X and, since $(s_n \circ h)$ converges, there is an x in X such that $a_n \to x$. By assumption we have that

$x \in M$ and consequently (s_n) converges.

$(ii) \Rightarrow (i)$: Take an arbitrary (x_n) in M and assume $x_n \to x$, for some $x \in X$. Hence we have a convergent sequence $s_n \circ h : \Omega(X) \longrightarrow \Omega(M) \longrightarrow \mathbf{2}$ and then, by assumption, (s_n) converges. Therefore, there exists $\overline{x} \in M$ such that $x_n \to \overline{x}$. Since X has unique sequential limits, we have $x = \overline{x}$.

$\square$

As pointed out previously, in the point-free setting convergence and clustering of a sequence coincide with that of its image factorisation, see Observation 2.3.9. This serves as motivation to examine sequences of subspaces in more detail: Consider a sequence of points (x_n) in a topological space X. For each $n \in \mathbb{N}$, the point x_n can be associated with the subspace $\{x_n\} \subseteq X$ which, in a natural way, gives rise to the sequence of subspaces $(\{x_n\})$. We put forward the following

Definition 3.1.5. A sequence $(\{x_n\})$ converges to x in X if for all $U \in \mathcal{U}_x$ there exists an $n \in \mathbb{N}$ such that $\{x_m\} \cap U \neq \emptyset$ (or equivalently, $\{x_m\} \subseteq U$) for all $m \geq n$.

Observation 3.1.6. We note that a sequence of points (x_n) converges to some x if and only if the sequence of subspaces $(\{x_n\})$ converges to x.

This observation serves as motivation to define convergence and clustering for arbitrary subspaces:

Definition 3.1.7. A sequence of subspaces (A_n) in X is *convergent* (resp. *strongly convergent*) if and only if there exists an $x \in X$ such that for every $U \in \mathcal{U}_x$ there is an $n \in \mathbb{N}$ with $A_m \cap U \neq \emptyset$ (resp. $A_m \subseteq U$) for all $m \geq n$.

Definition 3.1.8. A sequence of subspaces (A_n) in X *clusters* to some $x \in X$ if and only if for every $U \in \mathcal{U}_x$ and for every every $n \in \mathbb{N}$ there exists $m \geq n$ such that $A_m \cap U \neq \emptyset$.

We now have the following characterisation of convergence and clustering by means of open covers:

Proposition 3.1.9. *For an arbitrary sequence of subspaces (A_n) in a topological space X, the following statements are equivalent:*

(i) There exists $x \in X$ such that (A_n) converges (resp. strongly converges) to x.

(ii) For every open cover $\mathcal{C}$ of X there exists $U \in \mathcal{C}$ and $n \in \mathbb{N}$ such that $A_m \cap U \neq \emptyset$ (resp. $A_m \subseteq U$) for all $m \geq n$.

Proof. Similar to the proof of 2.3.1.

□

Proposition 3.1.10. *For an arbitrary sequence of subspaces (A_n) in a topological space X, the following statements are equivalent:*

(i) There exists $x \in X$ such that x is a cluster point of (A_n).

(ii) For every open cover $\mathcal{C}$ of X there exists $U \in \mathcal{C}$, and for all $n \in \mathbb{N}$ there exists $m \geq n$ such that $A_m \cap U \neq \emptyset$.

Proof. Similar to the proof of 2.3.1.

□

We now extend the classical definitions to take subspaces into account and introduce some new notation:

Definition 3.1.11. Let M denote a subspace of a topological space X. We say that M is

(a) Σ-*closed* (resp. Σ^s-*closed*) if any sequence of subspaces (A_n) of M which converges (resp. strongly converges) in X also converges (resp. strongly converges) in M.

(b) *sequentially compact* if every sequence of points in M contains a convergent subsequence.

(c) Σ-*compact* (resp. Σ^s-*compact*) if every sequence of subspaces of M has a convergent (resp. strongly convergent) subsequence.

Remarks 3.1.12. (i) A sequence of subspaces $(A_n) \subseteq M$ that converges (resp. strongly converges) in a subspace $M \subseteq X$, also converges (resp. strongly converges) in X.

(ii) Σ^s-compact spaces are also Σ-compact.

Proposition 3.1.13. *Let X be a topological space. Then*

(i) a closed subspace of X is Σ^s-closed.

(ii) a Σ^s-closed (resp. Σ-closed) subspace of X is sequentially closed.

Proof. (i) Assume $M \subseteq X$ is closed and take an arbitrary sequence of subspaces (A_n) on M. If (A_n) does not strongly converge in M, then there is an open cover $\mathcal{C}$ of M with for all $U \in \mathcal{C}$ and for all $n \in \mathbb{N}$ there exists $m \geq n$, $A_m \nsubseteq U$. Let $\mathcal{C}'$ be the open cover of X given by $\mathcal{C}' = \{U \cup (X \setminus M) \mid U \in \mathcal{C}\}$. Since $A_n \subseteq M$, for all n, it follows that (A_n) does not strongly converge in X.

(ii) This follows readily, since any sequence (a_n) can be considered as a sequence of subspaces $(\{a_n\})$.

□

Corollary 3.1.14. *For a sequential space X, the notions of closed, Σ^s-closed and sequentially closed subspace all coincide.*

Lemma 3.1.15. *For a topological space X,*

(i) if X is Σ^s-compact, then it is sequentially compact.

(ii) X is sequentially compact if and only if it is Σ-compact.

Proof. (i) Assume X is Σ^s-compact and let (a_n) be an arbitrary sequence. Then $(\{a_n\})$ is a sequence of subspaces of X, and hence has a strongly convergent subsequence $(\{a_{n_k}\})$. Then (a_{n_k}) is a convergent subsequence of (a_n).

(ii) $(\Rightarrow)$: Assume we have a sequentially compact space X and a sequence of subspaces (A_n). For every $n \in \mathbb{N}$, choose $a_n \in A_n$. Then, by assumption, (a_n) has a convergent subsequence (a_{n_k}). Thus, for every open cover $\mathcal{C}$ of X, there is a $U \in \mathcal{C}$ and $n \in \mathbb{N}$ such that $a_{n_k} \in U$, for all $k \geq n$. That is, $A_{n_k} \cap U \neq \emptyset$.

$(\Leftarrow)$: Similar proof to (i).

□

Remark 3.1.16. It is a well-known that fact that, in general, neither compactness nor sequential compactness implies the other. We can therefore conclude that neither compactness nor Σ-compactness implies the other as well.

If we view a space as a subspace of itself, as the author does, then we have

Proposition 3.1.17. *A Σ^s-compact space X is compact.*

Proof. Take an arbitrary open cover $\mathcal{C}$ of a Σ^s-compact space X and consider the constant subspace sequence (X_n), where $X_n = X$, for all $n \in \mathbb{N}$. By assumption there exists $U \in \mathcal{C}$ and $n \in \mathbb{N}$ such that $X_m \subseteq U$, for all $m \geq n$. Hence the open cover $\mathcal{C}$ has a finite subcover.

□

Observations 3.1.18. (i) This proposition highlights the fact that Σ^s-compactness is a very strong property and allows for the following characterisation by open covers: A space X is Σ^s-compact if and only if $X \in \mathcal{C}$ whenever $\mathcal{C}$ is an open cover of X.

(ii) If one considers only countable covers, then we have the following characterisation: For a topological space X, $X \in \mathcal{C}$ whenever $\mathcal{C}$ is a countable open cover of X if and only if every sequence in X converges.

(iii) We can show that a compact space does not, in general, satisfy these two properties above: Consider the set $X = \mathbb{N}$ with the co-finite topology. Then X is compact and the family $\mathcal{C} = \{(-\infty, n) \cup (n, \infty) \mid n \in \mathbb{N}\}$ is a countable open cover of X, but $X \notin \mathcal{C}$.

We now put forward the following point-free notions:

Definition 3.1.19. A sublocale $h : L \longrightarrow M$ is *sequentially closed* if for any sequence (s_n) on M, (s_n) is convergent whenever $(s_n \circ h)$ is convergent.

Definition 3.1.20. A frame L is

(a) *strongly sequentially compact* if $1 \in C$ whenever C is a cover of L.

(b) *sequentially compact* if any sequence on L has a convergent subsequence.

(c) *cluster compact* if any sequence on L clusters.

Remark 3.1.21. Frame homomorphisms $h : L \longrightarrow M$ trivially "extend" convergence in the sense that any convergent sequence on M will be convergent in L, since for any cover of C of L, $h(C)$ is a cover of M. Therefore, for a sublocale to be sequentially closed means that a sequence (s_n) on M converges if and only if $(s_n \circ h)$ converges.

The following set of results reveals the great extent to which classical notions transfer to the point-free setting:

Proposition 3.1.22. *(i) A closed sublocale is sequentially closed.*

(ii) If L is sequentially compact and $h : L \longrightarrow M$ a sequentially closed sublocale, then M is sequentially compact.

(iii) A sequentially compact frame is cluster compact.

(iv) If L is strongly sequentially compact and $k : N \longrightarrow L$ is injective, then N is strongly sequentially compact.

Proof. (i) Let $c \in L$, $\check{c} : L \longrightarrow \uparrow c$ be an arbitrary closed sublocale and s_n be a sequence on $\uparrow c$ such that $(s_n \circ \check{c})$ converges. Any cover C in $\uparrow c$ is also a cover in L, and hence there exists $a \in C$ and an $n \in \mathbb{N}$ such that for all $m \geq n$, $s_m(\check{c}(a)) = s_m(a) \neq 0$.

(ii) Take an arbitrary sequence (s_n) on M. Then $(s_n \circ h)$ is a sequence on L and hence has a convergent subsequence $(s_{n_k} \circ h)$. By assumption (s_{n_k}) converges.

(iii) Trivial.

(iv) Take an arbitrary cover C of N. Then $k(C)$ is a cover of L and by assumption, $1_L = k(1_N) \in k(C)$. Since k is injective we have that $1_N \in C$.

□

Remark 3.1.23. We note that if follows from Obs. 3.1.18 that if a frame L has the property that $1 \in C$ whenever C is a countable cover of L, then L is sequentially compact.

3.2 Countable compactness

Countable compactness is the first countable compactness notion to receive consideration due to its strong connection to sequences in classical topology. Recall that a family of subsets $\mathcal{F}$ of a topological space X is said to have the *finite intersection property* if $\mathcal{F} \neq \emptyset$ and $\bigcap \mathcal{G} \neq \emptyset$ for any finite $\mathcal{G} \subseteq \mathcal{F}$. The following result was taken from Engelking [33]:

Proposition 3.2.1. *A topological space X is compact if and only if every family of closed subsets which has the finite intersection property has non-empty intersection.*

Corollary 3.2.2. *A topological space X is countably compact if and only if every countable family of closed subsets which has the finite intersection property has non-empty intersection.*

The following well-known result characterises countable compactness in terms of sequential clustering:

Lemma 3.2.3. *A topological space X is countably compact if and only if any sequence (x_n) in X has a cluster point.*

Proof. ($\Rightarrow$): Assume X is countably compact and take an arbitrary sequence (x_n) in X. Define sets $F_n = \{x_m \,|\, m \geq n\}$ and set $B_n = \overline{F}_n$, for all $n \in \mathbb{N}$. Then clearly $\{B_n \,|\, n \in \mathbb{N}\}$ is a countable family of closed subsets of X that satisfies the finite intersection property. By assumption we have $x \in \bigcap_{n\in\mathbb{N}} B_n$, for some $x \in X$. Take arbitrary $U \in \mathcal{U}_x$ and $n \in \mathbb{N}$, then $U \cap F_n \neq \emptyset$ and there exists $m \geq n$ with $x_m \in U$.

($\Leftarrow$): Assume X is not countably compact, hence there is a countable family of closed subsets $\mathcal{F} = \{A_1, A_2, A_3, \ldots\}$ with the finite intersection property such that $\bigcap \mathcal{F} = \emptyset$. Put $F_n = \bigcap_{i=1}^{n} A_i$, then $\emptyset \neq F_n$ is closed for all $n \in \mathbb{N}$. Pick $x_n \in F_n$, then for all $x \in X$ there exists n such that $x \notin F_n$ and thus x is not a cluster point of (x_n).

$\square$

In the topological setting, sequences and related sequential notions are very useful for the characterising of countable compactness. We extend these results by taking sequences of subspaces into consideration:

Theorem 3.2.4. *The following are equivalent for a topological space X:*

(i) *X is countably compact.*

(ii) *Every sequence of points (a_n) in X clusters.*

(iii) *Every sequence of non-empty subspaces (A_n) of X clusters.*

(iv) *For any space Y, the projection $\pi_Y : X \times Y \longrightarrow Y$ maps closed subsets to sequentially closed subsets.*

(v) *For any space Y, the projection $\pi_Y : X \times Y \longrightarrow Y$ maps closed subsets to Σ^s-closed subsets.*

Proof. $(i) \Leftrightarrow (ii)$: See proof of Lemma 3.2.3.

$(ii) \Leftrightarrow (iii)$: Any sequence (a_n) can be viewed as a sequence of subspaces $(\{a_n\})$ and given a sequence of subspaces (A_n) of X, we can define a sequence (a_n) by choosing $a_n \in A_n$ for all $n \in \mathbb{N}$.

$(iii) \Rightarrow (v)$: Let $A \subseteq X \times Y$ be closed and consider the sequence of subspaces (A_n) in $\pi_Y(A)$ with (A_n) strongly convergent in Y. For $n \in \mathbb{N}$, define $B_n = \pi_X(\pi_Y^{-1}(A_n) \cap A)$, then $B_n \neq \emptyset$ for every n and (B_n) clusters in X.

Consider $y \in Y \backslash \pi_Y(A)$. Since A is closed, for all $x \in X$ there are open $U_x \subseteq X$ and $V_x \subseteq Y$ with $(x, y) \in U_x \times V_x$ and $U_x \times V_x \cap A = \emptyset$, giving an open cover $\mathcal{C}_y = \{U_x \,|\, x \in X\}$ of X. Since (B_n) clusters, we have $U_{x_y} \in \mathcal{C}_y$ such that for all $n \in \mathbb{N}$ there exists $m \geq n$, $B_m \cap U_{x_y} \neq \emptyset$, in which case $A_m \nsubseteq V_{x_y}$ (otherwise $\emptyset \neq (B_m \cap U_{x_y} \times A_m) \cap A \subseteq (U_{x_y} \times V_{x_y}) \cap A = \emptyset$).

Put $D_y = V_{x_y}$ and define $\mathcal{D} = \{D_y \,|\, y \in Y \backslash \pi_Y(A)\}$. Then $\mathcal{D}$ is an open cover of $Y \backslash \pi_Y(A)$ such that for all $D_y \in \mathcal{D}$ and for all $n \in \mathbb{N}$ there exists $m \geq n$, $A_m \nsubseteq D_y$. Let $\mathcal{C}$ be an open cover of $\pi_y(A)$, then $\mathcal{C} \cup \mathcal{D}$ is an open cover of Y. Since (A_n) is strongly convergent on Y, there is a $C \in \mathcal{C}$ and $n \in \mathbb{N}$ such that for all $m \geq n$, $A_m \subseteq C$. Thus (A_n) is strongly convergent on $\pi_y(A)$.

$(v) \Rightarrow (iv)$: Follows since Σ^s-closed implies sequentially closed.

$(iv) \Rightarrow (iii)$: Let (A_n) be a sequence in X and define $Y = \{1/n \,|\, n \in \mathbb{N}\} \cup \{0\}$, a subspace of $\mathbb{R}$. Let us define $A = \bigcup_{n\in\mathbb{N}} A_n \times \{1/n\}$, then $\overline{A}$ is closed in $X \times Y$. By assumption $\pi_Y(\overline{A})$ is sequentially closed and since $(1/n)$ is a sequence in $\pi_Y(\overline{A})$, we have that $0 \in \pi_Y(\overline{A})$.

Let $\mathcal{C}$ be an open cover of X, then there is $C \in \mathcal{C}$ such that $(C \times \{0\}) \cap \overline{A} \neq \emptyset$. Thus, for all $n \in \mathbb{N}$ we have

$$(C \times [0, 1/n)) \cap \left(\bigcup_{m\in\mathbb{N}} A_m \times \{1/m\} \right) \neq \emptyset,$$

hence there is $m \geq n$ such that $C \cap A_m \neq \emptyset$.

$\square$

Corollary 3.2.5. *If a topological space is sequentially compact, then it is countably compact. This implication follows from (ii) above.*

Next we characterise countably compact spaces by sequence *convergence*. This is achieved by requiring the open covers to have an additional property:

Definition 3.2.6. A family $\mathcal{F}$ of subsets of X is said to be *up-directed* if $U, V \in \mathcal{F}$ then there exists $W \in \mathcal{F}$ such that $U \subseteq W$ and $V \subseteq W$.

Observations 3.2.7. (i) Thus for a finite subset $\mathcal{G}$ of an up-directed family $\mathcal{F}$ of $\Omega(X)$, we can find a $W \in \mathcal{F}$ such that $\bigcup \mathcal{G} \subseteq W$.

(ii) Compact and countably compact spaces can easily be characterised by up-directed open covers: A topological space X is *compact* (resp. *countably compact*) if and only if $X \in \mathcal{C}$ whenever $\mathcal{C}$ is an up-directed (resp. up-directed and countable) open cover of X.

Proposition 3.2.8. *The following statements about a topological space X are equivalent:*

(i) X is countably compact.

(ii) Every sequence of points (a_n) converges for every up-directed and countable open cover of X.

(iii) Every sequence of subspaces (A_n) converges for every up-directed and countable open cover of X.

Proof. $(i) \Rightarrow (ii)$: This is obvious, since any up-directed and countable open cover contains X.

$(ii) \Rightarrow (iii)$: With similar reasoning as before, given a sequence of subspaces, we can construct a sequence of points by choosing a point from each subspace.

$(iii) \Rightarrow (i)$: Take an up-directed and countable open cover $\mathcal{C} = \{U_n \,|\, n \in \mathbb{N}\}$ of X. Assume $U_n \neq X$ for all $n \in \mathbb{N}$ and define a sequence (A_n) where $A_n = X \setminus \bigcup_{i=1}^{n} U_i$ for all $n \in \mathbb{N}$. By assumption there is a U_j and $n \in \mathbb{N}$ such that $A_m \cap U_j \neq \emptyset$ for all $m \geq n$. A contradiction, in any case, and hence we are done.

□

We now put forward the following definition and conclude this section with various characterisations of point-free countable compactness:

Definition 3.2.9. For a frame L, a subset $C \subseteq L$ is said to be *up-directed* if whenever $a, b \in C$ then there exists $c \in C$ such that $a \leq c$ and $b \leq c$.

Lemma 3.2.10. *If a sequence (s_n) on L does not cluster, then there is a cover $B = \{b_n \,|\, n \in \mathbb{N}\}$ of L with $b_n \leq b_{n+1}$ for every n and $s_m(b_n) = 0$ for all $m \geq n$.*

Proof. By definition, we have a cover C and for all $a \in C$ there exists $n \in \mathbb{N}$ such that $s_m(a) = 0$, for all $m \geq n$. Define the countable set $B = \{b_n \in L \,|\, n \in \mathbb{N}\}$ by

$$b_n = \bigvee \{a \in C \mid s_m(a) = 0, \ \forall m \geq n\}.$$

Clearly, B is a cover of L and $b_n \leq b_{n+1}$. Furthermore, for all $m \geq n$, $s_m(b_n) = 0$ since s_m is a frame homomorphism.

□

Theorem 3.2.11. *The following statements about a frame L are equivalent:*

(i) L is countably compact.

(ii) $1 \in C$ whenever C is an up-directed and countable cover of L.

(iii) Every prime upset converges for every countable cover of L.

(iv) Every sequence on L clusters (i.e. L is cluster compact.).

(v) Every sequence on L converges for every up-directed and countable cover of L.

Proof. $(i) \Leftrightarrow (ii)$: Obvious.

$(i) \Leftrightarrow (iii)$: [Countable version of the proof in [28]]. Assume L is countably compact and A a prime upset in L. Then any countable cover C will have a finite subcover, by assumption. The join of this finite subcover is in A, so A meets the subcover and therefore C.

Conversely, suppose each prime upset in L converges for every countable cover of L. Now take any countable cover C of L. If C has no finite subcover, then

$$A := \{x \in L \,|\, x \nleq \bigvee S \text{ for any finite } S \subseteq C\},$$

is a prime upset that misses C, contrary to the hypothesis. Consequently, every countable cover has a finite subcover.

$(i) \Rightarrow (iv)$: Assume there is a sequence on L that does not cluster. By Lemma 3.2.10, we have constructed a countable cover B with no finite subcover.

$(iv) \Rightarrow (i)$: Take a countable cover $A = \{a_n \in L \,|\, n \in \mathbb{N}\}$ and assume, w.l.o.g, that $a_n \leq a_{n+1}$. Assume $a_n \neq 1$ for all n and define the sequence $s_n : L \longrightarrow \uparrow a_n$ by $x \mapsto x \vee a_n$. By assumption then (s_n) clusters, hence there is $a_k \in A$ with for all $n \in \mathbb{N}$ there exists $m \geq n$ such that $s_m(a_k) = a_k \vee a_m \neq a_m$. If we choose $n = k$, then $s_m(a_k) = a_m$, resulting in a contradiction.

$(i) \Rightarrow (v)$: Take an arbitrary sequence (s_n) on L and an up-directed and countable cover $C = \{a_1, a_2, a_3, \ldots\} \subseteq L$. By assumption there is a finite subcover $C' \subseteq C$ and by up-directedness, $1 \in C$. Hence, there is an $a = 1 \in C$ and $n \in \mathbb{N}$ such that $s_m(a) \neq 0$, in fact $s_m(a) = 1$, for all $m \geq n$.

$(v) \Rightarrow (iv)$: Follows easily from Lemma 3.2.10, since B is an up-directed and countable cover.

□

Corollary 3.2.12. *If a frame is sequentially compact, then it is countably compact. This implication follows from* (iv) *above.*

3.3 Countable Almost Compactness

In the literature, almost compact spaces go by various names: H-closed [78], [60] and absolutely closed [22], to mention but two. In the countable case, Stone [71] called a countably almost compact space *feebly compact* and it is attributed to S. Mardešić and P. Papić. Other authors, Bagley et al. [5], Singal and Mathur [68] called such a space *lightly compact* and defined it as a space in which every locally finite family of open subsets of X is finite.

Porter and Woods provided the following characterisation of a feebly compact space in [61]:

Proposition 3.3.1. *The following are equivalent for a space* X*:*

(i) X *is feebly compact.*

(ii) If $\mathcal{C}$ *is a countable collection of open sets with the finite intersection property, then* $\bigcap\{\overline{C} \mid C \in \mathcal{C}\} \neq \emptyset$.

They also provided the following characterisation which, due to its open cover formulation, will serve as our definition:

Definition 3.3.2. A topological space X is *countably almost compact* if for every countable open cover $\mathcal{C}$ of X there exists a finite subfamily $\mathcal{A} \subseteq \mathcal{C}$, such that

$$X = \bigcup \{\overline{A} \mid A \in \mathcal{A}\}.$$

In Lemma 3.2.3 we have shown that a countably compact space X can be characterised by the clustering of any sequence in X. Next we show that countably almost compact spaces have a similar characterisation.

Definition 3.3.3. We say that x in a topological space X is a $**$-*cluster point* of a sequence (x_n) (resp. of subspaces (A_n)) in X if for every $U \in \mathcal{U}_x$ and for every $n \in \mathbb{N}$ there exists $m \geq n$ such that $x_m \in U^{**}$ (resp. $A_m \cap U^{**} \neq \emptyset$), where $U^* = X \backslash \overline{U}$. We will also say that (x_n) (resp. (A_n)) $**$-clusters in X.

Definition 3.3.4. An open cover $\mathcal{C}$ of X is said to be an open $**$-*cover* if whenever $U \in \mathcal{C}$, then $U^{**} \in \mathcal{C}$.

Observation 3.3.5. It is obvious that any sequence that clusters in X will also $**$-cluster.

We can now characterise countably almost compact spaces by means of $**$-clustering sequences:

Lemma 3.3.6. *A topological space X is countably almost compact if and only if any sequence (x_n) in X has a $**$-cluster point.*

Proof. ($\Rightarrow$): Assume X is countably almost compact and take an arbitrary sequence (x_n) in X. Define sets $A_n = \{x_m \mid m \geq n\}$ for all $n \in \mathbb{N}$. Then clearly A_n^{**} is a countable family of open subsets of X that satisfies the finite intersection property. By assumption we have $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n^{**}}$, for some $x \in X$. Take an arbitrary $U \in \mathcal{U}_x$ and $n \in \mathbb{N}$, then $U \cap A_n^{**} \neq \emptyset$. Hence $(U^{**} \cap A_n)^{**} \neq \emptyset$ and $U^{**} \cap A_n \neq \emptyset$ and thus there exists $m \geq n$ with $x_m \in U^{**}$.

($\Leftarrow$): Assume X is not countably almost compact. Then there exists a countable open cover $\mathcal{C} = \{U_n \mid n \in \mathbb{N}\}$ such that for all finite $\mathcal{A} \subseteq \mathcal{C}$ we have

$$X \neq \bigcup \{\overline{U} \mid U \in \mathcal{A}\}.$$

Set $V_n = U_1 \cup U_2 \cup \cdots \cup U_n$, for all $n \in \mathbb{N}$, then by assumption $\overline{V}_n \neq X$ and hence we can define a sequence (x_n) by choosing $x_n \in V_n^* \neq \emptyset$. Furthermore, for a fixed $n \in \mathbb{N}$, $x_m \notin V_n^{**}$ for all $m \geq n$. Then $\mathcal{C}' = \{V_n \,|\, n \in \mathbb{N}\}$ is an open cover such that for all $V \in \mathcal{C}'$ there exists an $n \in \mathbb{N}$ such that $x_m \notin V^{**}$ for all $m \geq n$.

$\square$

The definition above combined with Prop. 2.3.2 allows for the following characterisation via open covers:

Proposition 3.3.7. *For an arbitrary sequence (x_n) (resp. of subspaces (A_n)) in a topological space X, the following are equivalent:*

(i) There exists $x \in X$ such that x is a $**$-cluster point of (x_n) (resp. (A_n)).

(ii) For every open $**$-cover $\mathcal{C}$ of X there exists $U \in \mathcal{C}$ and for all $n \in \mathbb{N}$ there exists $m \geq n$ such that $x_m \in U$ (resp. $A_m \cap U \neq \emptyset$).

The following theorem highlights the fact that, with the correct open cover notion, we can characterise countably almost compact spaces via sequence and filter clustering:

Theorem 3.3.8. *The following statements about a topological space X are equivalent:*

(i) X is countably almost compact.

(ii) Every sequence of points (a_n) in X $**$-clusters.

(iii) Every sequence of subspaces (A_n) of X $**$-clusters.

(iv) Every countably based proper filter on X clusters.

Proof. $(i) \Rightarrow (ii)$: See proof of Lemma 3.3.6.

$(ii) \Rightarrow (iii)$: This follows readily, since we can choose $a_n \in A_n$ for every $n \in \mathbb{N}$.

(iii) $\Rightarrow$ *(iv)*: Assume a countably based proper filter $\mathcal{F}$, with basis $\{A_1, A_2, A_3 \ldots\}$, does not cluster. Then

$$\bigcup\{X \backslash \overline{F} \mid F \in \mathcal{F}\} = \bigcup\{F^* \mid F \in \mathcal{F}\} = X.$$

Define a $**$-cover $\mathcal{C} = \{F^* \mid F \in \mathcal{F}\}$ and sequence of subspaces (A_n), for all $n \in \mathbb{N}$. Assume that (A_n) form a decreasing chain and gake an arbitrary $U \in \mathcal{C}$, then $U = F^*$ for some $F \in \mathcal{F}$ and there exists $n \in \mathbb{N}$, such that $A_n \subseteq F$. Therefore $A_m \cap U = \emptyset$, for all $m \geq n$, and the sequence (A_n) does not cluster.

(iv) $\Rightarrow$ *(i)*: Assume X is not countably almost compact. Then there exists an open cover $\mathcal{C} = \{U_n \mid n \in \mathbb{N}\}$ such that for all finite $\mathcal{A} \subseteq \mathcal{C}$,

$$X \neq \bigcup\{\overline{A} \mid A \in \mathcal{A}\}.$$

Let $\mathcal{F}$ denote the filter with basis $\{U_n^* \mid n \in \mathbb{N}\}$. $\mathcal{F}$ is proper, since $U_n^* \neq \emptyset$ for all $n \in \mathbb{N}$. Moreover, $\bigcup\{F^* \mid F \in \mathcal{F}\} = X$, since $\mathcal{C}$ is a cover, and hence $\mathcal{F}$ does not cluster.

□

By employing the notion of up-directedness, which was introduced earlier, we can now define a countably almost compact space via open covers. Moreover, these spaces can easily be characterised by sequence convergence:

Proposition 3.3.9. *A topological space X is countably almost compact if and only if $X \in \mathcal{C}$ whenever $\mathcal{C}$ is an up-directed and countable open $**$-cover of X.*

Proof. $(\Rightarrow)$: Take an arbitary up-directed and countable open $**$-cover of X. By Def. 3.3.2 we have a finite $\mathcal{A} \subseteq \mathcal{C}$ such that

$$\bigcup_{A \in \mathcal{A}} \overline{A} = \overline{\bigcup_{A \in \mathcal{A}} A} = X.$$

By up-directedness, there exists an $U \in \mathcal{C}$ such that $\bigcup\{A \mid A \in \mathcal{A}\} \subseteq U$ and hence $\overline{U} = X$. Thus, $U^{**} = X$ and $U^{**} \in \mathcal{C}$ since $\mathcal{C}$ is a $**$-cover.

$(\Leftarrow)$: Let $\mathcal{C} = \{C_i \mid i \in \mathbb{N}\}$ be an arbitrary open cover of X, and put $D_n = \bigcup_{i=1}^{n} C_i$ for all $n \in \mathbb{N}$. Then $\mathcal{D} = \{D_n^{**} \mid n \in \mathbb{N}\}$ is an open and up-directed $**$-cover of X.

By assumption, $X \in \mathcal{D}$ and we have $C_1, C_2, \ldots, C_n$ such that $(\bigcup_{i=1}^{n} C_i)^{**} = X$. Hence

$$\overline{\bigcup_{i=1}^{n} C_i} = \bigcup_{i=1}^{n} \overline{C_i} = X$$

and X is countably almost compact.

□

Proposition 3.3.10. *The following statements about a topological space X are equivalent:*

(i) X is countably almost compact.

(ii) Every sequence of points (a_n) converges for every up-directed and countable open $**$-cover of X.

(iii) Every sequence of subspaces (A_n) converges for every up-directed and countable open $**$-cover of X.

We conclude this section with characterisations of countably almost compact frames afforded by results from literature and from the point setting above. A characterisation of a feebly compact space by Porter and Woods [61] allows for an easy point-free translation:

Definition 3.3.11. A frame L is *countably almost compact* if for any countable cover C of L, there is a finite subset $A \subseteq C$ such that $\bigvee A$ is dense, that is $(\bigvee A)^{**} = 1$.

We have previously characterised countable compactness by convergence and clustering. We now look towards finding such characterisations for countable almost compactness.

Recall that a proper filter $F \subseteq L$ clusters in a frame L if $\mathrm{sec}F \cap C \neq \emptyset$ (see Def. 2.4.6), for every cover C of L. Hong [47] proved that this can be stated equivalently as $\bigvee\{x^* \mid x \in F\} \neq 1$. Filter clustering can characterise countably almost compact frames:

Lemma 3.3.12 (Banaschewski *et al.* [11])**.** *For any frame L, every countably based proper filter clusters if and only if L is countably almost compact.*

Proof. ($\Rightarrow$): Consider $\bigvee_n a_n = 1$, where $a_n \leq a_{n+1}$ and assume $a_n^{**} \neq 1$ for all $n \in \mathbb{N}$, that is, L is not countably almost compact. Then the filter F with basis $\{a_n^* \in L \,|\, n \in \mathbb{N}\}$ does not cluster, since $\bigvee_n a_n^{**} = 1$.

($\Leftarrow$): Take a countably based proper filter F with basis $a_1 \geq a_2 \geq \ldots$. If $\bigvee_n a_n^* = 1$, then, by countable almost compactness, $a_k^{**} = 0$ and hence $a_k = 0$ for some k, a contradiction. Thus $\bigvee_n a_n^* \neq 1$, saying F clusters.

□

The following two definitions are the point-free counterpart of earlier pointset definitions and we add one more to completely capture all the nuances required by the point-free setting.

Definition 3.3.13. A sequence (s_n) on a frame L $**$-*clusters* if for every $**$-cover C of L there exists $a \in C$ such that for all $n \in \mathbb{N}$ there exists $m \geq n$ with $s_m(a) \neq 0$.

Definition 3.3.14. A subset A of a frame L is said to be $**$-*regular* if whenever $a^{**} \in A$, then $a \in A$.

Theorem 3.3.15. *The following statements about a frame L are equivalent:*

(i) *L is countably almost compact.*

(ii) *$1 \in C$ whenever C is an up-directed and countable $**$-cover of L.*

(iii) *Every $**$-regular and prime upset converges for every countable cover of L.*

(iv) *Every countably based proper filter clusters in L.*

(v) *Every sequence on L converges for every up-directed and countable $**$-cover of L.*

Proof. $(i) \Leftrightarrow (ii)$: Obvious.

$(i) \Leftrightarrow (iii)$:[A modification of the proof in [28]] Assume L is countably almost compact and A a $**$-regular and prime upset in L. Then, by assumption, any countable cover C will have a finite dense subset. The join of this finite dense subset is in A, since A is an up-set and $**$-regular, so A meets the subset and therefore C.

Conversely, assume every $**$-regular and prime upset converges for every countable cover and consider an arbitrary countable cover C_L of L. If C_L has no finite subset whose join is dense in L, define

$$F = \{x \in L \,|\, x \nleq (\bigvee A)^{**} \text{ for any finite } A \subseteq C_L\}.$$

One easily shows that F is a prime upset. Furthermore, if $a \notin F$, then $a \leq (\bigvee A)^{**}$ for some finite $A \subseteq C_L$. Furthermore, $a^{**} \leq (\bigvee A)^{****} = (\bigvee A)^{**}$ and consequently $a^{**} \notin F$.

Consequently, F is a $**$-regular and prime upset that misses C_L, contradicting our initial hypothesis.

$(i) \Leftrightarrow (iv)$: See proof of Lemma 3.3.12.

$(ii) \Rightarrow (v)$: Trivial.

$(v) \Rightarrow (i)$: Take an arbitrary countable cover $C = \{a_1, a_2, a_3, \dots\}$ of L and define an up-directed and countable $**$-cover $C^{**} = \{b_1^{**}, b_2^{**}, b_3^{**}, \dots\}$ by setting $b_n = a_1 \vee a_2 \vee \cdots \vee a_n$ for all $n \in \mathbb{N}$. Assume $b_n^{**} \neq 1$ for all n, otherwise we are done, and define a sequence $s_n : L \longrightarrow \uparrow b_n^{**}$ given by $s_n(x) = x \vee b_n^{**}$.

By assumption there is a $b_k^{**} \in C^{**}$ and a $n \in \mathbb{N}$ such that $s_m(b_k^{**}) \neq 0$ for all $m \geq n$. If $k < n$, then $s_m(b_k^{**}) = b_k^{**} \vee b_m^{**} = b_m^{**} = 0$ (the bottom element of $\uparrow b_m^{**}$). If $k \geq n$, then choose $m \geq k$ which also gives $s_m(b_k^{**}) = 0$. A contradiction either way, and hence there is an l such that $b_l^{**} = 1$.

□

Corollary 3.3.16. *If every sequence on L $**$-clusters, then L is countably almost compact.*

Proof. Assume L is not countably almost compact, then by Thm. 3.3.15 there is a proper filter F with basis $\{a_1, a_2, a_3, \ldots\}$ that does not cluster in L. Then we have a cover C of L for which $\mathrm{sec}F \cap C = \emptyset$.

Define a $**$-cover $C^{**} = \{a^{**} \,|\, a \in C\}$ and set $b_n = a_1 \wedge a_2 \wedge \ldots \wedge a_n$ for all $n \in \mathbb{N}$. We have that for all n, $b_n \neq 0$ since $a_i \in F$ for all $i \in \mathbb{N}$. Now define a sequence (s_n) on L by setting

$$s_n(x) = (x \mapsto x \wedge b_n) : L \longrightarrow {\downarrow} b_n.$$

We will show that this sequence does not $**$-cluster: Take an arbitrary $x \in C^{**}$, then $x = a^{**}$ for some $a \in C$. Since $\mathrm{sec}F \cap C = \emptyset$, we have $a \wedge a_n = 0$, for some $a_n \in F$, and $(a \wedge a_n)^{**} = a^{**} \wedge a_n^{**} = 0$ and hence $a^{**} \wedge a_n = 0$. Moreover, $a^{**} \wedge b_m = s_m(a^{**}) = 0$ for all $m \geq n$.

□

Corollary 3.3.17. *If a frame is countably compact, then it is countably almost compact.*

Remark 3.3.18. We say a cover C of a frame L is *co-regular* if for every $a \in C$ there exist $b \in C$ such that $a \prec b$. With the next section in mind, we highlight the role that co-regular covers play:

Corollary 3.3.19. *If a frame L is countably almost compact, then it implies each of the following conditions:*

(i) Every countable co-regular cover of L has a finite subcover.

(ii) $1 \in C$ whenever C is an up-directed, countable and co-regular cover of L.

(iii) Every prime upset converges for every countable co-regular cover of L.

(iv) Every sequence on L converges for every up-directed, countable and co-regular cover of L.

3.4 Pseudocompactness

If X is a topological space, then $C(X)$ denotes the space of all continuous real-valued functions $f : X \longrightarrow \mathbb{R}$. In the case where only the bounded continuous functions are considered, we denote the family of functions by $C^*(X)$. A topological space is said to be *pseudocompact* if $C(X) = C^*(X)$. This definition was first given by Hewitt [41] for Tychonoff spaces, that is, completely regular T_1-spaces.

In the point-free setting, pseudocompactness has received considerable attention. Gilmour and Banaschewski [10] characterised pseudocompactness in terms of the co-zero part of a frame and Dube and Matutu [29] provided both external and internal characterisations. These characterisations were achieved without imposing complete regularity on the frame. Earlier characterisations, within the context of completely regular frames, can be found in [16] and [77].

Recall that a frame L is said to be *pseudocompact* if every continuous real function $\alpha : \mathfrak{L}(\mathbb{R}) \longrightarrow L$ is bounded, and by [10], this holds if and only if any sequence (a_n) in L such that

$$\bigvee a_n = 1 \text{ and } a_n \prec\!\prec a_{n+1} \text{ for all } n \in \mathbb{N}$$

terminates, that is, $a_k = 1$ for some k. A recent article by Banaschewski *et al.* [11] characterised pseudocompactness in terms of countable almost compactness with complete regularity imposed:

A completely regular frame L is pseudocompact if and only if it is countably almost compact.

Remark 3.4.1. In fact, one can show that a countably almost compact frame is pseudocompact. Complete regularity needs only be imposed for the forward implication.

However, a more favourable characterisation, due to its connection to covers and then by extension to sequences, can be found in [29]. The authors, Dube and Matutu, introduced an additional notion on the covers of the frame:

Definition 3.4.2. A cover C of a frame L is *co-completely regular* if for each $a \in C$ there exists $b \in C$ such that $a \prec\prec b$.

Definition 3.4.3. A filter base F in a frame L is said to be *completely regular* if for each $a \in F$ there exists $b \in F$ such that $b \prec\prec a$.

Lemma 3.4.4 (Dube, Matutu [29])**.** *The following statements are equivalent for any frame L:*

(i) L is pseudocompact.

(ii) Every countable co-completely regular cover of L has a finite subcover.

(iii) Every countable completely regular based proper filter clusters in L.

Proof. See proof in [29].

□

Theorem 3.4.5. *The following statements are equivalent for a frame L:*

(i) L is pseudocompact.

(ii) $1 \in C$ whenever C is an up-directed, countable and co-completely regular cover of L.

(iii) Every prime upset in L converges for every countable co-completely regular cover of L.

(iv) Every countable completely regular based proper filter clusters in L.

(v) Every sequence on L converges with respect to every up-directed, countable and co-completely regular cover of L.

Proof. $(i) \Leftrightarrow (ii)$: Follows from Lemma 3.4.4.

$(ii) \Rightarrow (iii)$: Take an arbitrary prime upset F in L and co-completely regular cover $C = \{a_n \,|\, n \in \mathbb{N}\}$ of L. Set $a'_n = \bigvee_{1 \le k \le n} a_k$, then by assumption there exists an m such that $\bigvee_{1 \le k \le m} a_k = 1$. Since F is a prime upset, there is a k such that $a_k \in F$, and hence $F \cap C \neq \emptyset$.

$(iii) \Rightarrow (ii)$:[A modification of the proof in [28]] Take an arbitrary up-directed, countable and co-completely regular cover C of L and assume every prime upset converges with respect to it. If $1 \notin C$, then define

$$F = \{x \in L \,|\, x \nleq a \text{ for any } a \in C\}.$$

It is easy to show that F is an upset and indeed prime. But F misses C which contradicts our initial assumption.

$(i) \Leftrightarrow (iv)$: See Lemma 3.4.4.

$(ii) \Rightarrow (v)$: Trivial.

$(v) \Rightarrow (ii)$: Take an arbitrary up-directed and co-completely regular cover $C = \{a_n \,|\, n \in \mathbb{N}\}$ of L. If $a_n \neq 1$ for all $n \in \mathbb{N}$, define a sequence $s_n : L \longrightarrow \uparrow a_n$ by $s_n(x) = x \vee a_n$. By assumption, there is a $a_k \in C$ and $n \in \mathbb{N}$ such that $s_m(a_k) \neq 0 = a_m$ for all $m \geq n$. With the same reasoning as in Thm. 3.3.15 we are left with a contradiction. Therefore there is a k such that $a_k = 1$.

□

Corollary 3.4.6. *If a frame L is countably almost compact, then it is pseudocompact.*

Observation 3.4.7. We note that conditions (ii) and (iii) are equivalent regardless of the countable and co-completely regular properties imposed on the covers. In general, where α denotes a collection of properties imposed on the cover, we have:

$1 \in C$ whenever C is an up-directed α-cover of L if and only if every prime upset in L converges for every α-cover of L.

We conclude this chapter with a few final remarks:

(i) If we compare the proofs throughout this chapter it is evident that, by imposing additional properties on the covers, we could characterise a weaker compactness notion, in many cases, with a similar characterisation of a stronger compactness notion. Moreover, the proofs of the characterisations of weaker compactness notions require, in many cases, small modifications to the proofs of the characterisations of the stronger compactness notions.

(ii) It is clear that the convergence/clustering of point-free sequences can successfully characterise some countable compact notions, and thereby extending their usefulness. An interesting, but yet still unanswered, question is the role that point-free sequences and convergence/clustering plays in the metric frame setting.

(iii) In the point-free setting, the cover characterisation of the compactness notions considered in this chapter is summarised in the diagram below. Underneath each compact notion, indicated within the parenthesis, are the cover properties necessary to characterise the specific compactness notion. The formulation is as follow:

A frame L is $\langle$insert compact notion$\rangle$ if and only if $1 \in C$ whenever C is a(n) $\langle$insert cover properties$\rangle$ cover of L.

Furthermore, the compactness notions have been ordered such that any specific notion is implied by the notion(s) higher up in the diagram.

$$\begin{tikzcd}
& \begin{array}{c}\textit{strongly sequentially compact}\\ \textit{(}-\textit{)}\end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}\textit{compact}\\ \textit{(up-directed)}\end{array} \arrow[dr] & & \textit{sequentially compact} \arrow[dl] \\
& \begin{array}{c}\textit{countably compact}\\ \textit{(up-directed, countable)}\end{array} \arrow[d] & \\
& \begin{array}{c}\textit{countably almost compact}\\ \textit{(up-directed, countable, }{**}\textit{-cover)}\end{array} \arrow[d] & \\
& \begin{array}{c}\textit{pseudocompact}\\ \textit{(up-directed, countable, co-completely regular)}\end{array} &
\end{tikzcd}$$

Chapter 4

Compact Boolean frames

The well-known Kuratowski-Mrówka theorem states that a space X is compact if and only if the second projection $\pi_2 : X \times Y \longrightarrow Y$ is a closed map, for every space Y. This characterisation allows for a point-free version, as given by Pultr and Tozzi in [63]. Dube [28] continued on this work and proved that a frame L is compact if and only if the image of a nearly closed sublocale under the coproduct injection $\iota_K : K \longrightarrow L \oplus K$ is nearly closed, for every frame K.

In this chapter we will recall extension-closed and nearly closed sublocales and give a point-free characterisation in terms of upset convergence. Our aim is to provide a characterisation of compact Boolean frames, similar to the Kuratowski-Mrówka type theorem in the point-free setting.

4.1 Extension-closed and nearly closed sublocales

Let A be a subspace of a topological space X and $h : \Omega(X) \longrightarrow \Omega(A)$ the sublocale map given by $U \mapsto U \cap A$. Harris [36] defines an *extension-closed* subspace as one in which every open cover of the subspace extends to an open cover of the entire space. This choice in terminology is apt and follows from the classic proof of

If a space X is compact and $A \subseteq X$ closed, then A is compact.

The proof requires the "extension" of a cover of A to a cover of the whole space X. Dube [28] used the following as motivation for a point-free definition:

A is extension-closed if and only if for each open cover $\mathcal{C}$ of A there exists an open cover $\mathcal{D}$ of X such that $\{h(D) \,|\, D \in \mathcal{D}\} = \mathcal{C}$.

Definition 4.1.1. A sublocale $h : L \longrightarrow M$ is *extension-closed* if for every cover C_M of M there is a cover C_L of L such that $h(C_L) = C_M$.

As convergence is an important notion in the study of sequences, we are interested in characterisations that render such a notion. Hong [47] was first to introduce convergence for filters in frames. Dube [28] showed that one can characterise an extension-closed sublocale in terms of upset convergence. He also makes the following observation:

Observation 4.1.2. A sublocale $h : L \longrightarrow M$ is extension-closed if and only if $h_*(C_M)$ is a cover of L for each cover C_M of M.

A notion closely related to extension-closedness is that of nearly closedness. Herr–lich introduces a pointset definition in [39], which coincides for Hausdorff spaces with that of closed subspaces, from which Dube removed all references to points in [28]:

Definition 4.1.3. A sublocale $h : L \longrightarrow M$ is *nearly closed* if for every cover C_M of M there is a cover C_L of L such that for each $a \in C_L$ there is a finite $A \subseteq C_M$ with $h(a) \leq \bigvee A$.

Examples 4.1.4. (i) Any closed sublocale is extension-closed.

(ii) For every frame L, the co-diagonal map $\nabla : L \oplus L \longrightarrow L$ where $\nabla(a \oplus b) = a \wedge b$, is extension-closed. Any cover $C_L = \{a_i \,|\, i \in I\}$ of L extends to a cover $C_{L\oplus L} = \{(1 \oplus a_i) \,|\, i \in I\}$ of $L \oplus L$. Moreover,

$$\bigvee_{i \in I}(1 \oplus a_i) = 1 \oplus \bigvee_{i \in I} a_i = 1 \oplus 1 \text{ and } \nabla(C_{L\oplus L}) = C_L.$$

(iii) It is easy to show that an extension-closed sublocale is nearly closed.

Definition 4.1.5. A sublocale $h : L \longrightarrow M$ is *up-directed extension-closed* if for every up-directed cover C_M of M there is a cover C_L of L such that $h(C_L) = C_M$.

Observation 4.1.6. A sublocale $h : L \longrightarrow M$ is up-directed extension-closed if and only if $h_*(C_M)$ is a cover of L for each up-directed cover C_M of M.

Lemma 4.1.7. *A sublocale $h : L \longrightarrow M$ is nearly closed if and only if for every up-directed cover C_M of M, $C_L = \{a \in L \,|\, \exists\, c \in C_M$ such that $h(a) \leq c\}$ is a cover of L.*

Proof. ($\Rightarrow$): Assume h is nearly closed and C_M an arbitrary up-directed cover. Then by definition there is a cover $\overline{C}_L$ such that for all $a \in \overline{C}_L$ there is a finite $A \subseteq C_M$ with $h(a) \leq \bigvee A$. But since C_M is up-directed, there is a $c \in C_M$ such that $\bigvee A \leq c$. Then $\overline{C}_L \subseteq C_L$ and $\bigvee C_L = 1$.

($\Leftarrow$): Let C_M be an arbitrary cover of M. Then $\overline{C}_M = \{\bigvee A \,|\, A \text{ finite}, A \subseteq C\}$ is up-directed and from assumption we have that $C_L = \{a \in L \,|\, \exists\, c \in \overline{C}_M, h(a) \leq c\}$ is a cover of L, and we are done.

$\square$

Dube stops short from making the following observation in [28]:

Proposition 4.1.8. *A sublocale $h : L \longrightarrow M$ is nearly closed if and only if it is up-directed extension-closed.*

Proof. ($\Rightarrow$): Take an arbitrary up-directed cover C_M of M. Then, by assumption, $C_L = \{a \in L \,|\, \exists\, c \in C_M$ such that $h(a) \leq c\}$ is a cover of L and for each $a \in C_L$, $a \leq h_*(c)$ for some $c \in C_M$. Consequently, $h_*(C_M)$ is a cover of L and h is up-directed extension-closed.

($\Leftarrow$): Take an arbitrary up-directed cover C_M of M. Then $h_*(C_M) \subseteq C_L = \{a \in L \,|\, \exists\, c \in C_M$ such that $h(a) \leq c\}$ and, by assumption, $\bigvee h_*(C_M) = 1_L$. By Lemma 4.1.7, h is nearly closed.

$\square$

A nearly closed subspace is then one for which an up-directed cover of the subspace extends to the entire space. We extend the definition of upset convergence to take into account up-directed covers:

Definition 4.1.9. An upset F in frame L is said to be *up-directed convergent* if for any up-directed cover C of L, $F \cap C \neq \emptyset$.

In [36] Harris shows that an extension-closed subspace can also be characterised by filterbase convergence:

A subspace of a topological space is extension-closed if and only if every filterbase on the subspace that converges in the containing space also converges in the subspace.

Dube provides a point-free characterisation of this result in [28], which we combine with Prop. 4.1.8 to obtain a characterisation of a nearly closed sublocale in terms of upset convergence. The proof is a slight modification to the one in [28].

Proposition 4.1.10. *A sublocale $h : L \to M$ is nearly closed if and only if an upset F is up-directed convergent in M whenever $h^{-1}(F)$ converges in L.*

Proof. ($\Rightarrow$): Assume $h : L \longrightarrow M$ is nearly closed and $h^{-1}(F)$ converges on L. If C is an arbitrary up-directed cover of M, then by 4.1.6 and 4.1.8 we have that $h_*(C)$ covers L. Since $h^{-1}(F) \cap h_*(C) \neq \emptyset$, we can take $c \in h^{-1}(F) \cap h_*(C)$. Then $h(c) \in F$ and $c = h_*(d)$ for some $d \in C$. Furthermore, $h(c) = h(h_*(d)) = d$ (h is onto) and hence $d \in \uparrow F = F$ and moreover $d \in F \cap C$.

($\Leftarrow$): Assume $h : L \longrightarrow M$ is not nearly closed, then by 4.1.6 and 4.1.8 there exists an up-directed cover C of M such that a $:= \bigvee h_*(C) \neq 1_L$. Define $F = h(L \setminus \downarrow a)$ $\neq \emptyset$. Then F satisfies the following conditions:

(i) F is an upset: Take $b \in L \setminus \downarrow a$ and $y \in M$ with $h(b) \leq y$. Then $b \leq h_*(y) \in L \setminus \downarrow a$ and therefore $y = h(h_*(y)) \in F$.

(ii) $h^{-1}(F)$ is convergent: If D covers L, then $D \not\subseteq \downarrow a$ and thus $D \cap (L \setminus \downarrow a)$ is non-empty. But $L \setminus \downarrow a \subseteq h^{-1}(h(L \setminus \downarrow a)) = h^{-1}(F)$ giving $D \cap h^{-1}(F) \neq \emptyset$.

(iii) F is not up-directed convergent: If there is $x \in F \cap C$, then $x = h(b)$ for some $b \in L \setminus \downarrow a$ and $h(b) \in C$. Since $b \leq h_*(h(b))$, we have that $h_*(h(b)) \in L \setminus \downarrow a$. But $a = \bigvee h_*(C) \geq h_*(h(b)) \geq b$. This contradicts the assumption that $b \in L \setminus \downarrow a$ and therefore $F \cap C = \emptyset$.

□

Observation 4.1.11. From the results above we note that we do not require up-directed covers in L, although they are essential in M. In fact, one can easily show that if C is an up-directed set in M, then $h_*(C)$ is an up-directed set in L, that is, a sublocale $h : L \longrightarrow M$ "reflects" up-directedness.

4.2 Almost closed sublocales

Pultr and Tozzi provided a point-free version of the famous Kuratowski-Mrówka theorem of topology in [63]. Dube [28] applied his research on nearly closed sublocales to binary co-products in frames and gave a characterisation of compact frames. We extend the research of these authors to render a Kuratowski-Mrówka type characterisation for compact Boolean frames.

Definition 4.2.1. A sublocale $h : L \longrightarrow M$ is called *almost closed* if for each cover C_M of M there is a cover C_L of L such that for each $a \in C_L$ there is a finite $A \subseteq C_M$ with $h(a^*) \geq (\bigvee A)^*$.

Remark 4.2.2. In the case that $M \subseteq L$, pseudocomplementation taken in M and L may not necessarily coincide. Consequently, subscripts denoting the particular frame are added to the notation.

Examples 4.2.3. (i) The co-diagonal map $\nabla : L \oplus L \longrightarrow L$ is almost closed, since $(1 \oplus a)^* = 1 \oplus a^*$, for $a \in L$. (See Lemma 4.3.6.)

(ii) If M is finite, then any sublocale $h : L \longrightarrow M$ is almost closed.

Observation 4.2.4. If $h : L \longrightarrow M$ is almost closed and M regular, then h is nearly closed.

Proof. Take an arbitrary cover C_M of a regular frame M. We write $C_M = \{c_i \,|\, i \in I\}$ and because M is regular we have $c_i = \bigvee\{x \in M \,|\, x \prec c_i\}$, for all $i \in I$. Define $\overline{C}_M = \{\, x \in M \,|\, x \prec c_i \text{ for some } i \in I\,\}$ which is then clearly a cover. By assumption there is a cover C_L of L such that for each $a \in C_L$ there is a finite $A \subseteq \overline{C}_M$ with $h(a^*) \geq (\bigvee A)^*$.

We note that for each $x \in A$ there is an $i \in I$ such that $x \prec c_i$. Fix $a \in C_L$ and take a finite $A \subseteq \overline{C}_M$ with $h(a^*) \geq (\bigvee A)^*$. Then there exists a finite $K \subseteq I$ such that $\bigvee A \prec \bigvee\limits_{i\in K} c_i$ and hence $(\bigvee A)^{**} \prec \bigvee\limits_{i\in K} c_i$ (see Preliminaries). Moreover,

$$h(a) \leq h(a)^{**} \leq h(a^*)^* \leq (\bigvee A)^{**} \leq \bigvee_{i\in K} c_i.$$

$\square$

Mirrored on the definition of a co-completely regular cover, see Dube and Matutu [29], we now recall the following definition:

Definition 4.2.5. A cover C of a frame L is said to be *co-regular* if for each $a \in C$ there exists $b \in C$ such that $a \prec b$.

We extend our definition for a nearly-closed sublocale and almost closed sublocale by replacing the arbitrary cover of M with a co-regular one:

Definition 4.2.6. A sublocale $h : L \longrightarrow M$ is called *co-regular* nearly closed (resp. almost closed) if for each co-regular cover C_M of M there is a cover C_L of L such that for each $a \in C_L$ there is a finite $A \subseteq C_M$ with $h(a) \leq \bigvee A$ (resp. $h(a^*) \geq (\bigvee A)^*$).

Focusing on co-regular covers, we note that an almost closed sublocale is nearly closed. The proof is similar to the proof of Observation 4.2.4:

Observation 4.2.7. If sublocale $h : L \longrightarrow M$ is co-regular almost closed, then h is co-regular nearly closed.

Next we turn our attention to some of the inheriting properties of almost closed sublocales:

Proposition 4.2.8. *(i) Almost compact sublocales are almost closed.*

(ii) If L is almost compact and a sublocale $h : L \longrightarrow M$ is almost closed, then M is almost compact.

(iii) L is almost compact if and only if $h : N \longrightarrow L$ is almost closed for every sublocale h.

Proof. (i) Consider a sublocale $h : L \longrightarrow M$ with almost compact M. Take an arbitrary cover C_M of M. By assumption there is a finite subset $A \subseteq C_M$ with $(\bigvee A)^{**} = 1$. Any cover C_L of L will do.

(ii) Assume L is almost compact and the sublocale $h : L \longrightarrow M$ is almost closed. Take an arbitrary cover C_M of M and, since h is almost closed, associated cover C_L of L such that

$$\forall a \in C_L, \exists A \subseteq C_M \, \text{finite} : h(a^*) \geq (\bigvee A)^*.$$

Since L is almost compact, there is a finite $\overline{A} \subseteq C_L$ with $(\bigvee \overline{A})^{**} = 1$. We write $\overline{A} = \{a_k \,|\, k \in K\}$ with K finite. Furthermore, for each $k \in K$ there is a finite $C_k \subseteq C_M$ with $h(a_k^*) \geq (\bigvee C_k)^*$, since h is almost closed. Moreover,

$$\begin{aligned} 0_M &= h(0_L) = h((\bigvee \overline{A})^*) = h(\bigwedge_{k \in K} a_k^*) = \bigwedge_{k \in K} h(a_k^*) \\ &\geq \bigwedge_{k \in K} (\bigvee C_k)^* = \Big(\bigvee_{k \in K} (\bigvee C_k)\Big)^* \\ &= \Big(\bigvee (\bigcup_{k \in K} C_k)\Big)^* \end{aligned}$$

Put $B := \bigcup_{k \in K} C_k$, then $B \subseteq C_M$ and B is finite with $(\bigvee B)^{**} = 1$.

(iii) $(\Rightarrow)$: Assume L is almost compact and $h : N \longrightarrow L$ a sublocale. Then by (i), h is almost closed.
$(\Leftarrow)$: Consider the (almost) compact frame $\mathfrak{J}L$ and sublocale $v_L : \mathfrak{J}L \longrightarrow L$ (see Preliminaries). By assumption we have v_L is almost closed and then by (ii) it follows that L is almost compact.

$\square$

4.3 Kuratowski-Mrówka-type characterisation of compact Boolean frames

We apply the theory developed above on almost closed sublocales to characterize compact Boolean frames in a similar manner as Dube did in [28]. Let $p_L : L \times M \longrightarrow L$ and $p_M : L \times M \longrightarrow M$ denote the projection maps.

Definition 4.3.1. We say a frame L satisfies the *finite unit decomposition property* (briefly, UD_{fin}) if for each frame M and each decomposition of the unit

$$1_{L\oplus M} = \bigvee\{\, a_i \oplus b_i \,|\, i \in I \,\}$$

we have

$$1_M = \bigvee\left\{ \bigwedge\{b_i \,|\, i \in K\,\} \,|\, \text{ finite } K \subseteq I \text{ such that } \bigvee\{a_i \,|\, i \in K\,\} = 1_L \right\}.$$

We recall the following important and well-known fact, proved by Pultr and Tozzi, without proof:

Proposition 4.3.2 (Pultr, Tozzi [63])**.** *Each compact frame L satisfies UD_{fin}.*

Remark 4.3.3. Chen [23] proves this result in a different way without requiring the Axiom of Choice.

Lemma 4.3.4. *For any frame K and Boolean frame L, an almost closed sublocale $h : K \longrightarrow L$ is nearly closed.*

Proof. Take an arbitrary cover C_L of L. By assumption there is a cover C_K of K such that for each $a \in C_K$ there is a finite $A \subseteq C_L$ with $h(a^*) \geq (\bigvee A)^*$. Furthermore, for each $a \in C_K$, with A chosen as above,

$$h(a) \leq h(a)^{**} \leq h(a^*)^* \leq (\bigvee A)^{**} = \bigvee A \,(\text{since } L \text{ is Boolean}).$$

□

We recall the following Lemma proved by Dube in [28], and include the proof from that paper for completeness.

Lemma 4.3.5 (Dube [28]). *If a frame L is compact and a sublocale $h : L \longrightarrow M$ is nearly closed, then frame M is compact.*

Proof. Assume we have a compact L and nearly closed $h : L \longrightarrow M$. For an arbitrary cover C_M of M there is a cover C_L of L such that for each $b \in L$ there is a finite $A_b \subseteq C_M$ such that $h(b) \leq \bigvee A_b$. Let $\{c_1, c_2, \ldots, c_n\}$ be the finite subcover of C_L. Then we have

$$\begin{aligned} 1_M &= h(c_1) \vee h(c_2) \vee \ldots \vee h(c_n) \\ &\leq \bigvee A_{c_1} \vee \bigvee A_{c_2} \vee \ldots \vee \bigvee A_{c_n} \\ &\leq \bigvee \bigcup_n A_{c_n}. \end{aligned}$$

□

Lemma 4.3.6. *Let L and M be frames. Then for each $b \in M$,*

$$1 \oplus b^* = (1 \oplus b)^*.$$

Proof. $(\leq)$: Consider the coproduct injection $\iota_M : M \longrightarrow L \oplus M$ given by $\iota_M(a) = 1 \oplus a$, which is a frame homomorphism, and hence

$$1 \oplus b^* = \iota_M(b^*) \leq \iota_M(b)^* = (1 \oplus b)^*.$$

$(\geq)$: By definition,

$$\begin{aligned} (1 \oplus b)^* &= \bigvee \{u \in L \oplus M \,|\, (1 \oplus b) \wedge u = \mathbf{0}\} \\ &= \bigvee \{x \oplus y \,|\, x \in L, y \in M \text{ such that } (1 \oplus b) \wedge (x \oplus y) = \mathbf{0}\} \\ &= \bigvee \{x \oplus y \,|\, x \in L, y \in M \text{ such that } x \oplus (b \wedge y) = \mathbf{0}\}. \end{aligned}$$

Now take arbitrary and fixed $c \in L$, $d \in M$ with $c \oplus (b \wedge d) = \mathbf{0}$. We will prove that $c \oplus d \leq 1 \oplus b^*$:

Case 1: If $c = 0$, then clearly $c \oplus d = \mathbf{0} \oplus d = \mathbf{0} \leq 1 \oplus b^*$.
Case 2: If $c \neq 0$, then $b \wedge d = 0$ (else $c \oplus (b \wedge d) \neq \mathbf{0}$) and hence $d \leq b^*$. Moreover,

$$c \oplus d = \downarrow(c, d) \cup \mathbf{0} \leq \downarrow(1, b^*) \cup \mathbf{0} = 1 \oplus b^*.$$

□

We now state the main result of this section. The proof of the following theorem is based on a similar proof in [28].

Theorem 4.3.7. *The following statements about a Boolean frame L are equivalent:*

(i) L is compact.

(ii) L satisfies UD_{fin}.

(iii) For every M, the image of an almost closed sublocale under the coproduct injection $\iota_M : M \longrightarrow L \oplus M$ is an almost closed sublocale.

Proof. $(i) \Rightarrow (ii)$: See Proposition 4.3.2, due to Pultr and Tozzi [63].

$(ii) \Rightarrow (iii)$: Assume L satisfies UD_{fin} and consider an almost closed sublocale $h : L \oplus M \longrightarrow N$ and injection $\iota_M : M \longrightarrow L \oplus M$. Let $l : M \longrightarrow h\iota_M(M)$ be the sublocale given by $l(x) = h(\iota_M(x)) = h(1 \oplus x)$.

Take an arbitrary cover C of $h\iota_M(M)$. Then C is a cover of N and hence, by assumption, there is a cover $C_{L\oplus M}$ of $L \oplus M$ such that for each $u \in C_{L\oplus M}$ there is a finite $A_u \subseteq C$ with $h(u^*) \geq (\bigvee A_u)^*$.

Define

$$\begin{aligned} \overline{C} &= \{\, a \oplus b \,|\, (a, b) \in u \text{ for some } u \in C_{L\oplus M} \,\} \\ &= \{\, a \oplus b \,|\, a \oplus b \leq u \text{ for some } u \in C_{L\oplus M} \,\}, \end{aligned}$$

then obviously, since $u = \bigvee\{a \oplus b \,|\, a \oplus b \leq u\}$ for all $u \in C_{L\oplus M}$, $\overline{C}$ is a cover of $L \oplus M$. Since L satisfies UD_{fin}, we have

$$1_M = \bigvee\{\, \bigwedge p_M(K) \,|\, K \text{ finite}, K \subseteq \bigcup\{u \,|\, u \in C\} \text{ such that } \bigvee p_L(K) = 1_L \,\}.$$

Put $C_M := \{\, \bigwedge p_M(K) \,|\, K \text{ finite}, K \subseteq \bigcup\{u \,|\, u \in C\} \text{ such that } \bigvee p_L(K) = 1_L \,\}$. Fix an arbitrary element of C_M, that is, $K = \{\, (a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k) \,\}$ and $u_1, u_2, \ldots, u_k \in C_{L\oplus M}$ $(k \geq 1)$ such that $a_i \oplus b_i \leq u_i$, for all $1 \leq i \leq k$ and

$a_1 \vee a_2 \vee \ldots \vee a_k = 1_L$.

We then have the following string of equalities,

$$\begin{aligned} l((\bigwedge p_M(K))^*) &= h(\iota_M((b_1 \wedge b_2 \wedge \ldots \wedge b_k)^*)) \\ &= h(1_L \oplus (b_1 \wedge b_2 \wedge \ldots \wedge b_k)^*) \\ &= h((1_L \oplus (b_1 \wedge b_2 \wedge \ldots \wedge b_k))^*) \text{ (by 4.3.6)} \end{aligned}$$

If we put $b := b_1 \wedge b_2 \wedge \ldots \wedge b_k$, we have (see Preliminaries)

$$1_L \oplus (b_1 \wedge b_2 \wedge \ldots \wedge b_k) = (a_1 \vee a_2 \vee \ldots \vee a_k) \oplus b = (a_1 \oplus b) \vee (a_2 \oplus b) \vee \ldots \vee (a_k \oplus b).$$

Now, for each $1 \leq i \leq k$ we have $b \leq b_i$ and

$$a_i \oplus b = \downarrow(a_i, b) \cup \mathbf{0} \subseteq \downarrow(a_i, b_i) \cup \mathbf{0} = a_i \oplus b_i,$$

and hence

$$1_L \oplus (b_1 \wedge b_2 \wedge \ldots \wedge b_k) \leq (a_1 \oplus b_1) \vee (a_2 \oplus b_2) \vee \ldots \vee (a_k \oplus b_k) \leq u_1 \vee u_2 \vee \ldots \vee u_k.$$

By taking pseudocomplements we then have

$$(1_L \oplus (b_1 \wedge b_2 \wedge \ldots \wedge b_k))^* \geq (u_1 \vee u_2 \vee \ldots \vee u_k)^* = u_1^* \wedge u_2^* \wedge \ldots \wedge u_k^*.$$

Therefore,

$$\begin{aligned} l((\bigwedge p_M(K))^*) &\geq h(u_1^* \wedge u_2^* \wedge \ldots \wedge u_k^*) \\ &= h(u_1^*) \wedge h(u_2^*) \wedge \ldots \wedge h(u_k^*) \\ &\geq (\bigvee A_{u_1})^* \wedge (\bigvee A_{u_2})^* \wedge \ldots \wedge (\bigvee A_{u_k})^* \\ &= (\bigvee A_{u_1} \vee \bigvee A_{u_2} \vee \ldots \vee \bigvee A_{u_k})^* \\ &\geq (\bigvee \bigcup_{j=1}^{k} A_{u_j})^*, \end{aligned}$$

and $\bigcup_{j=1}^{k} A_{u_j}$ is finite.

$(iii) \Rightarrow (i)$: We are going to express L as a nearly closed sublocale of a compact frame, and the result will then follow from Lemma 4.3.5.

Consider the (dense) sublocale $v_L : \mathfrak{J}L \longrightarrow L$, co-diagonal map $\Delta : L \oplus L \longrightarrow L$ and (unique) frame homomorphism $id_L \oplus v_L : L \oplus \mathfrak{J}L \longrightarrow L \oplus L$ given by $(id_L \oplus v_L)(a \oplus I) = a \oplus v_L(I) = a \oplus (\bigvee I)$. Define then $k = \Delta \circ (id \oplus v_L) : L \oplus \mathfrak{J}L \longrightarrow L$ determined by $k(a \oplus I) = \Delta(a \oplus (\bigvee I)) = a \wedge (\bigvee I)$, for $I \in \mathfrak{J}L$ and $a \in L$.

We will show that k is almost closed:
Take an arbitrary cover C_L of L. Then $\{\, c \oplus L \,|\, c \in C_L \,\}$ is a cover of $L \oplus \mathfrak{J}L$ and for any $c \in C_L$,

$$k((c \oplus L)^*) = k(c^* \oplus L) = c^* \wedge \bigvee L = c^* \geq (\bigvee \{c\})^*,$$

and $\{c\} \subseteq C_L$ is finite. By assumption, the image of this sublocale under the injection $i_{\mathfrak{J}L} : \mathfrak{J}L \longrightarrow L \oplus \mathfrak{J}L$ is almost closed, that is, the sublocale $l : \mathfrak{J}L \longrightarrow (k \circ i_{\mathfrak{J}L})(\mathfrak{J}L)$, given by $l(I) = k(i_{\mathfrak{J}L}(I))$, is almost closed.

Furthermore, we claim $(k \circ i_{\mathfrak{J}L})(\mathfrak{J}L) = L$:
Proof. ($\subseteq$:) Obvious.
($\supseteq$:) Take arbitrary $a \in L$, then $\downarrow a \in \mathfrak{J}L$ and

$$(k \circ i_{\mathfrak{J}L})(\downarrow a) = k(i_{\mathfrak{J}L}(\downarrow a)) = k(1_L \oplus \downarrow a) = 1_L \wedge \bigvee \downarrow a = 1_L \wedge a = a.$$

Thus we have an almost closed sublocale $l : \mathfrak{J}L \longrightarrow L$ and Boolean L, therefore Lemma 4.3.4 yields that l is nearly closed. Finally we observe that $l : \mathfrak{J}L \longrightarrow L$ is nearly closed and $\mathfrak{J}L$ compact, then by 4.3.5 we have that L is compact.

$\square$

Chapter 5

Applications of generalised points and filters

In this chapter we will briefly touch on some topics afforded by the generalised notions of filter, point and convergence. The purpose is not to present an exhaustive inquiry into each topic, but rather to initiate and demonstrate future research possibilities.

5.1 Convergence-closedness

Let L and M denote frames. All filters are to be read as proper. We call any $(0, \wedge, 1)$-homomorphism $f : L \longrightarrow M$ a *generalised filter* of L. The set of all generalised filters of L with codomain M, respectively all generalised points of L with codomain M, will be denoted by $\mathrm{Fil}_M(L)$, respectively $\mathrm{Pt}_M(L)$.

Definition 5.1.1. We say a generalised filter $f \in \mathrm{Fil}_M(L)$ *converges* to a generalised point $h \in \mathrm{Pt}_M(L)$ if $h \leq f$ (pointwise order) and we denote it by

$$f \underset{M}{\longrightarrow} h.$$

M is to be thought of as carrying possible truth values, and for $M = \mathbf{2}$, the concepts of generalised filter, respectively generalised point, reduce to the classical ones of filter, respectively point.

Consider the diagram below where $S \subseteq L$ is a sublocale set with corresponding sublocale homomorphism $\nu_S : L \longrightarrow S$ where $\nu_S(a) = \bigwedge\{\, s \in S \,|\, a \leq s\,\}$, $f \in \mathrm{Fil}_M(S)$ and $h \in \mathrm{Pt}_M(L)$.

$$\begin{tikzcd}
L \arrow[r, "\nu_S"] \arrow[d, "h"'] & S \arrow[r, "f"] \arrow[dl, dotted, "\overline{h}"] & M \\
M & &
\end{tikzcd}$$

Figure 5.1: Convergence closed

We now propose the following definition:

Definition 5.1.2. For a frame L and sublocale set $S \subseteq L$, we say S is *convergence-closed* if it satisfies the following condition: $\forall\ M \in \mathbf{Frm}$, $\forall\ f \in \mathrm{Fil}_M(S)$, $\forall\ h \in \mathrm{Pt}_M(L)$ we have

$$\text{If } f\nu_S \underset{M}{\longrightarrow} h, \text{ then there exists } \overline{h} : S \longrightarrow M \text{ such that } h = \overline{h}\nu_S.$$

We immediately note that $\overline{h} \in \mathrm{Pt}_M(S)$ and that f converges to $\overline{h}$:

Lemma 5.1.3. *Let L and M be frames, $S \subseteq L$ a sublocale set that is convergence-closed, $f \in Fil_M(S)$ and $h \in Pt_M(L)$ such that $f\nu_S \underset{M}{\longrightarrow} h$. Then, as per 5.1.2, $\overline{h} : S \longrightarrow M$ is a frame homomorphism with $\overline{h} \leq f$.*

Proof. Clearly, $\overline{h}(1) = \overline{h}(\nu_S(1)) = h(1) = 1$ and with $c := \bigwedge S$, $\overline{h}(c) = \overline{h}(\nu_S(0)) = h(0) = 0$. For every $s, t \in S$ we have

$$\begin{aligned}
\overline{h}(s \wedge t) &= \overline{h}(\nu_S(s) \wedge \nu_S(t)) = \overline{h}(\nu_S(s \wedge t)) \\
&= h(s) \wedge h(t) = \overline{h}(\nu_S(s)) \wedge \overline{h}(\nu_S(t)) \\
&= \overline{h}(s) \wedge \overline{h}(t).
\end{aligned}$$

For any $T \subseteq S$, we have that

$$\begin{aligned}
\overline{h}(\bigvee T) &= \overline{h}(\nu_S(\bigvee T)) = h(\bigvee T) = \bigvee\{h(t) \,|\, t \in T\} \\
&= \bigvee\{\overline{h}(\nu_S(t)) \,|\, t \in T\} = \bigvee\{\overline{h}(t) \,|\, t \in T\},
\end{aligned}$$

and hence $\overline{h}$ is a frame homomorphism. To conclude, note that for all $s \in S$,

$$\overline{h}(s) = \overline{h}(\nu_S(s)) = h(s) \leq f(\nu_S(s)) = f(s).$$

□

In a topological space X, a subspace $A \subseteq X$ is closed if and only if A contains all limit points of filters in X that contain A. Here an analogous result is true in the point-free setting:

Proposition 5.1.4. *Let L be a frame and $S \subseteq L$ be a sublocale set. Then the following assertions are equivalent:*

(i) S is convergence-closed,

(ii) S is closed.

Proof. $(ii) \Rightarrow (i)$: Assume $S = {\uparrow}c$ for some $c \in L$ and hence $\nu_S = (\cdot) \vee c$. Furthermore, let M denote a frame together with a $(0, \wedge, 1)$-homomorphism $f : {\uparrow}c \longrightarrow M$ and a frame homomorphism $h : L \longrightarrow M$ such that $h \leq f((\cdot) \vee c)$. Then, because

$$h(c) \leq f(c \vee c) = f(c) = 0,$$

we have $h(c) = 0$ and hence the restriction $h|_{\uparrow c} : {\uparrow}c \longrightarrow M$ of h to ${\uparrow}c$ satisfies $h|_{\uparrow c} \circ \nu_S = h$. Therefore we have S is convergence-closed.

$(i) \Rightarrow (ii)$: Assume that S is convergence-closed and put $c := \bigwedge S$. Then $S \subseteq {\uparrow}c$ and we are done if we can show that ${\uparrow}c \subseteq S$. Fix $b \in L$ with $c \leq b$. Consider the frame homomorphism $h := (\cdot) \vee b : L \longrightarrow {\uparrow}b$, and put $f := (\cdot) \vee b : S \longrightarrow {\uparrow}b$. Then $f(c) = c \vee b = b$ and $f(1) = 1 \vee b = 1$. Also, for every $s, t \in S$ we have

$$f(s \wedge t) = (s \wedge t) \vee b = (s \vee b) \wedge (t \vee b) = f(s) \wedge f(t).$$

So f is a $(0, \wedge, 1)$-homomorphism and because, for all $x \in L$, $f(\nu_S(x)) = \nu_S(x) \vee b \geq x \vee b = h(b)$, we have that $f\nu_S \underset{{\uparrow}b}{\longrightarrow} h$. By the convergence-closedness of S there exists a map $\overline{h} : S \longrightarrow {\uparrow}b$ with $h = \overline{h}\nu_S$. Then it follows that

$$b = h(b) = \overline{h}(\nu_S(b)) = \overline{h}(\nu_S(\nu_S(b)) = h(\nu_S(b)) = \nu_S(b) \vee b = \nu_S(b),$$

and hence $b \in S$.

□

5.2 α-Hausdorff property

For a topological space X, a classical characterisation of the Hausdorff separation axiom (or T_2-axiom) is that the diagonal $\{(x,x) \mid x \in X\}$ is closed in the product $X \times X$. By imitating this result in the point-free setting Isbell [48] provided a frame definition:

A frame L is *I(sbell)-Hausdorff* (*strongly-Hausdorff* in [48]) if the codiagonal

$$\Delta : L \oplus L \longrightarrow L,$$

given by $\Delta(u) = \bigvee\{x \wedge y \mid (x,y) \in u\} = \bigvee\{x \mid (x,x) \in u\}$, is a closed sublocale.

This is not an extension of the classical result, since in general $\Omega(X \times X)$ is not isomorphic to $\Omega(X) \oplus \Omega(X)$. Another classical characterisation states that every filter on a Hausdorff space X has at most one limit point. By employing generalised filters and generalised points we can mimic this result in **Frm**:

Definition 5.2.1. A frame L is said to be *α-Hausdorff* if a generalised filter $f : L \longrightarrow M$ converges to at most one generalised point of L (with codomain M), that is, if we have generalised points $h_1, h_2 : L \longrightarrow M$ with $h_1 \leq f$ and $h_2 \leq f$, then $h_1 = h_2$.

Proposition 5.2.2. *If a topological space X is sober and $\Omega(X)$ is α-Hausdorff, then X is Hausdorff.*

Proof. Assume open neighbourhood filters $\mathcal{U}_{x_1}$, $\mathcal{U}_{x_2} \subseteq \mathcal{F}$ for some $x_1, x_2 \in X$ and filter $\mathcal{F}$ on X. Define a $(0,1,\wedge)$-homomorphism $f : \Omega(X) \longrightarrow \mathbf{2}$ by setting $f(U) = 1$ if and only if $U \in \mathcal{F}$:

Clearly $f(X) = 1$ and $f(\emptyset) = 0$. If $U_1, U_2 \in \mathcal{F}$ then $U_1 \cap U_2 \in \mathcal{F}$. If $U_1 \notin \mathcal{F}$ or $U_2 \notin \mathcal{F}$, then $U_1 \cap U_2 \notin \mathcal{F}$. In any case,

$$f(U_1 \cap U_2) = f(U_1) \wedge f(U_2).$$

Next, define frame homomorphisms $h_i : \Omega(X) \longrightarrow \mathbf{2}$ by setting $h_i(U) = 1$ if and only if $U \in \mathcal{U}_{x_i}$, for $i = 1, 2$. We now show that $h_i \leq f$:

For any $U \in \Omega(X)$, if $U \in \mathcal{F}$, then $f(U) = 1$ and we have $h_i(U) \leq f(U)$. On the other hand, if $U \notin \mathcal{F}$, then $U \notin \mathcal{U}_{x_i}$ and $h_i(U) = f(U) = 0$.

By assumption we have that $h_1 = h_2$. As a result $\mathcal{U}_{x_1} = \mathcal{U}_{x_2}$ and by sobriety we conclude that $x_1 = x_2$.

$\square$

α-Hausdorff frames exhibit some of the inheriting properties that Hausdorff spaces posses:

Proposition 5.2.3. *If L is α-Hausdorff and $s : L \longrightarrow M$ a sublocale, then M is α-Hausdorff.*

Proof. Take a generalised filter $f : M \longrightarrow N$ on M and assume it converges to two generalised points $h_1, h_2 : M \longrightarrow N$, that is, $h_1 \leq f$ and $h_2 \leq f$ (see figure below).

$$\begin{tikzcd} L \arrow[r, two heads, "s"] & M \arrow[r, "f"] \arrow[d, bend right, "h_1"'] \arrow[d, bend left, "h_2"] & N \\ & N & \end{tikzcd}$$

Now consider the composite maps $f \circ s : L \longrightarrow N$ and $h_i \circ s : L \longrightarrow M$, for $i = 1, 2$. Then $f \circ s$ is a $(0, 1, \wedge)$-homomorphism, the h_i are frame homomorphisms and we have $h_i \circ s \leq f \circ s$, for $i = 1, 2$.

$$\begin{tikzcd} L \arrow[r, "f \circ s"] \arrow[d, bend right, "h_1 \circ s"'] \arrow[d, bend left, "h_2 \circ s"] & N \\ N & \end{tikzcd}$$

By the α-Hausdorff property we have that $h_1 \circ s = h_2 \circ s$. Since s is onto (right cancelable) we conclude this proof with $h_1 = h_2$.

$\square$

Corollary 5.2.4. *If $L \oplus L$ is α-Hausdorff, then L is α-Hausdorff.*

Johnstone [50] defined a point-free (locale) notion that mimicked the T_1-axiom in topology. He called this notion the T_U-axiom (T_U for "totally unordered") and continued by proving that an I-Hausdorff (strongly Hausdorff) locale satisfies the T_U-axiom. This result also holds true for α-Hausdorff frames:

Definition 5.2.5. We say a frame L satisfies the T_U*-axiom* if for any pair of frame maps $f, g : L \longrightarrow M$,

$$f \leq g \text{ implies } f = g.$$

Proposition 5.2.6. *An α-Hausdorff frame satisfies the T_U-axiom.*

Proof. For an arbitrary frame L and any pair of frame maps $f, g : L \longrightarrow M$, assume $f \leq g$. Then trivially g is a $(0, 1, \wedge)$-homomorphism, $f \leq g$ and $g \leq g$. By assumption it follows that $f = g$.

□

By employing the theory of the downset-functor $\mathfrak{D} : \mathbf{SLat} \longrightarrow \mathbf{Frm}$, see section 1.11.1, and by replacing L with its free frame $\mathfrak{D}(L)$, we can also prove the converse of Prop 5.2.6:

Proposition 5.2.7. *For a frame L, if $\mathfrak{D}L$ satisfies the T_U-axiom, then L is α-Hausdorff.*

Proof. Assume we have a $(0, 1, \wedge)$-homomorphism $f : L \longrightarrow M$ and frame homomorphisms $h_1, h_2 : L \longrightarrow M$ with $h_i \leq f$, for $i = 1, 2$. Then as per Prop. 1.11.1 we have uniquely corresponding frame homomorphisms $f', h_1', h_2' : \mathfrak{D}L \longrightarrow M$.

$$\begin{tikzcd}
\mathfrak{D}L \arrow[r, "f'"] \arrow[d, bend right, "h_1'"'] \arrow[d, bend left, "h_2'"] & M \\
M &
\end{tikzcd}$$

Furthermore, for any $A \in \mathfrak{D}L$, $h_1'(A) = \bigvee\{h_1(a) \mid a \in A\} \leq \bigvee\{f(a) \mid a \in A\} = f'(A)$, since $h_1 \leq f$. Similarly we have that $h_2'(A) \leq f'(A)$, for all $A \in \mathfrak{D}L$. By assumption $\mathfrak{D}L$ satisfies the T_U-axiom and $h_i' \leq f'$, and therefore $h_1' = h_2'$.

For any $a \in L$, we have $h_1(a) = h_1'(\downarrow a) = h_2'(\downarrow a) = h_2(a)$. Hence $h_1 = h_2$.

$\square$

Corollary 5.2.8. *For any frame L, if $\mathfrak{D}L$ is α-Hausdorff, then L is α-Hausdorff.*

5.3 Point-free countability properties

Replacing points of a space $x \in X$ with generalised points of a frame, we can easily provide point-free notions for first and second countability:

Definition 5.3.1. A frame L is said to be

(a) *first countable* if for every generalised point $h : L \to M$ there is a countable $A \subseteq \{x \in L \,|\, h(x) \neq 0\} := F$ such that for every $x \in F$ there exists $a \in A$ with $a \leq x$.

(b) *second countable* if it has a countable base $B \subseteq L$, that is, for every $a \in L$ there exists $B' \subseteq B$ such that $\bigvee B' = a$.

Proposition 5.3.2. *Every second countable frame L is first countable.*

Proof. Given an arbitrary point of a frame $h : L \longrightarrow M$ and define $F = \{x \in L \,|\, h(x) \neq 0\}$. By assumption we have a countable base $B \subseteq L$. Define $B' = \{b \in B \,|\, h(b) \neq 0\} \subseteq F$, which is clearly countable.

Take an arbitrary $x \in F$, then there is a $B_x \subseteq B$ such that $\bigvee B_x = x$. Furthermore, $\bigvee h(B_x) = h(\bigvee B_x) = h(x) \neq 0$, and therefore there exists $b \in B_x$ such that $h(b) \neq 0$. Ultimately, $b \leq x$ and $b \in B'$.

$\square$

Next we turn to separability. Dube was the first (to our knowledge) to have provided a point-free version of it in [27]. We start out with a rephrasing of the classical definition by means of (pointed) sequences, which we then in turn generalise for the point-free setting:

Proposition 5.3.3. *Let X denote a sober topological space. The following statements are equivalent:*

(i) X is separable.

(ii) There is a sequence $s_n : \Omega(X) \longrightarrow \mathbf{2}$ such that for every non-empty $U \in \Omega(X)$ there exists $k \in \mathbb{N}$, $s_k(U) = 1$.

Proof. $(i) \Rightarrow (ii)$: Assume we have a countable $D = \{x_1, x_2, x_3, \ldots\}$ with $\overline{D} = X$, that is, for every $x \in X$ and for every $U \in \Omega(X)$, $x \in U \Rightarrow U \cap D \neq \emptyset$. Define a sequence $s_n : \Omega(X) \longrightarrow \mathbf{2}$ by setting $s_n(U) = 1 \Leftrightarrow x_n \in U$. Take an arbitrary non-empty $U' \in \Omega(X)$, then for every $x \in U'$ we have $U' \cap D \neq \emptyset$. Hence there is a $k \in \mathbb{N}$ such that $x_k \in U'$ and $s_k(U') = 1$.

$(ii) \Rightarrow (i)$: Assume we have a sequence $s_n : \Omega(X) \longrightarrow \mathbf{2}$ and take an arbitrary $x \in X$ and an arbitrary $U \in \mathcal{U}_x$. By sobriety, we can define a sequence (x_n) and hence a countable $D = \{x_1, x_2, x_3, \ldots\}$. By assumption there exists $k \in \mathbb{N}$ such that $s_k(U) = 1 \Leftrightarrow x_k \in U$ and ultimately $U \cap D \neq \emptyset$.

□

This serves as motivation for the following:

Definition 5.3.4. A frame L is said to be *weakly separable* if there exists a sequence $s_n : L \to M_n$ such that for every $a \in L \backslash \{0\}$ there exists $k \in \mathbb{N}$ with $s_k(a) = 1$.

Observation 5.3.5. Dube [27] defined a separable frame L as one that has a countable set $S \subseteq L \backslash \{1\}$ such that for every $a \in L \backslash \{0\}$ there exists $s \in S$, $a \vee s = 1$. One easily shows that this notion is stronger than weak separability:

Proposition 5.3.6. *If a frame L is separable, then it is weakly separable.*

Proof. Given a separable frame L with countable set $S = \{x_1, x_2, x_3, \ldots\}$. Define a sequence $s_n : L \longrightarrow \uparrow x_n$ by setting $s_n(a) = a \vee x_n$. Then for any $a \in L \backslash \{0\}$, there exists a $k \in \mathbb{N}$ such that $1 = a \vee x_k = s_k(a)$.

□

Proposition 5.3.7. *A Lindelöf metric frame (L, d) is weakly separable.*

Proof. For any $n \in \mathbb{N}$ we have covers $\mathcal{U}^d_{1/n} = \{a \in L \,|\, d(a) < 1/n\}$. By assumption there is a countable subcover $A_n \subseteq \mathcal{U}^d_{1/n}$, for every $n \in \mathbb{N}$. Let A' denote the countable union of the A_n. Define a sequence $s_n : L \longrightarrow \downarrow x_n$ by setting $s_n(a) = a \wedge x_n$, where $x_n \in A'$ for $n \in \mathbb{N}$. Choose arbitrary $x \in L\backslash\{0\}$. Then there is $y \in L$ such that y is uniformly below x, that is, there exists $\epsilon > 0$ and a cover U^d_ϵ such that $U^d_\epsilon y \leq x$.

Furthermore, there is $m \in \mathbb{N}$ such that $m > \frac{1}{\epsilon}$ and hence $\frac{1}{m} < \epsilon$. Consequently, $A_m \subseteq \mathcal{U}^d_{1/m} \subseteq U^d_\epsilon$, and since A_m is a cover, there is $z \in A_m$, $z \wedge y \neq 0$ and thus $z \leq U^d_\epsilon y \leq x$. Let $z = x_j \in A_m$, since x_j is the top element in $\downarrow x_j$, we have $1 = x_j \wedge x_j = z \wedge x_j \leq x \wedge x_j = s_j(x)$.

$\square$

We conclude this chapter with some final remarks:

(i) This chapter serves to initiate further research into the generalised notions, as introduced in this dissertation, by considering a few varied applications. First we show that with the notions of generalised point and generalised filter, we can provide a point-free analogue of a classical characterisation of a closed subspace.

(ii) Next we turn to mimicking the Hausdorff separation axiom in frames. This has been done before by authors such as Isbell [48]and, Dowker and Strauss [25]. The connection between α-Hausdorff and other point-free Hausdorff properties, as well as its interplay with other separation axioms, will form part of subsequent investigations.

(iii) Looking ahead to research into metric frames, we conclude by touching on point-free countability properties.

Chapter 6

Dini and Stone-Weierstrass Properties in point-free topology

The fundamental Dini's Theorem, named after the Italian mathematician Ulisse Dini (1845–1918), is one of few situations in mathematics where, subject to additional conditions, uniform convergence is implied by pointwise convergence. This result has had far-reaching applications throughout topology and analysis (e.g. in the theory of ordinary differential equations). To be precise, for a topological space X, Dini's Theorem states:

If X is compact (Hausdorff), then every monotone sequence $(f_n)_n$ of continuous real functions which converges pointwisely to some continuous real function f, converges uniformly to f.

On the other hand, relabeling the conclusion of Dini's Theorem and calling it the *Dini Property*, one may quite naturally ask which spaces X satisfy the

Dini Property: Every sequence of monotone continuous real functions $(f_n)_n$ on X which converges pointwisely to some continuous real function f on X, converges uniformly to f.

To that end, a great deal of attention has been paid throughout the literature to find exactly such spaces. It is clear that all compact (Hausdorff) spaces satisfy the Dini Property. Other answers or partial answers to this question have been available in the literature and we will mention a few that are relevant. Concerning the Dini Property, it was proved in Colmez [24] and Glicksberg [35] that a Tychonoff space satisfies the Dini Property if and only if it is pseudocompact. Among more recent related work, we mention Kundu and Raha [53], where the authors observed that the Tychonoff condition is not essential, that is, a space satisfies the Dini Property if and only if it is pseudocompact. They also studied the Strong Dini Property (where instead of sequences, one considers arbitrary nets) and showed that a completely regular space satisfies the Strong Dini Property if and only if it is compact.

Another fundamental theorem about the ring $C(X)$ of continuous real-valued functions on a compact Hausdorff space, with far reaching applications in analysis is the celebrated Stone-Weierstrass theorem. The first form, dealing with uniform approximation of continuous real-valued functions on a compact real interval by polynomial functions, was proved by Karl Weierstrass in 1885 and this theorem was later generalized by Marshall H. Stone in 1937 to the following form:

For X compact Hausdorff, every point-separating (unital) $\mathbb{R}$-subalgebra (i.e. a subalgebra containing the constant functions) of $C(X)$ is dense in $C(X)$ with respect to the topology of uniform convergence.

Just as with Dini's theorem, one can now naturally ask which spaces X satisfy the

Stone Weierstrass Property: Every point-separating (unital) $\mathbb{R}$-subalgebra of $C^*(X)$ is dense in $C^*(X)$ with respect to the topology of uniform convergence.

Here $C^*(X)$ stands for the subring of $C(X)$ consisting of all bounded continuous real-valued functions. It was proved in Hewitt [40] that for Tychonoff spaces, the Stone-Weierstrass property is equivalent to compactness. In Kundu and Raha [53] the authors derive, combining their results with older results from Stephenson [70] and Raha and Srivastava [65], that for completely Hausdorff spaces (i.e. spaces

where $C(X)$, or equivalently $C^*(X)$, separates points), the Strong Dini Property and the Stone-Weierstrass Property are equivalent, and also equivalent to the condition that every cover of X consisting of cozero sets has a finite subcover.
It is the aim of this chapter to investigate Dini-type properties as well as the Stone-Weierstrass property in the pointfree setting, that is for frames. It must be mentioned that what we call the Dini Property here is called the Weak Dini Property in [53]. In the context of pointfree topology, a Stone-Weierstrass Theorem has been proved by B. Banaschewski (see B. BANASCHEWSKI [8, 9]).

6.1 On the Dini Property and the Strong Dini Property

For a net $(\alpha_\eta)_{\eta\in D}$, we will denote the preorder on D by $\leq$. We implicitly assume that this indexing partially ordered set D is upward-directed in the sense that for all $\eta_1, \eta_2 \in D$, there exists $\xi \in D$ such that $\eta_1 \leq \xi$ and $\eta_2 \leq \xi$. If $D := \mathbb{N}$ (the set of positive integers, excluding 0) with the natural order, the corresponding net is called a *sequence*, in which case we will denote it by $(\alpha_n)_n$. We say that a net $(\alpha_\eta)_{\eta\in D}$ in a given ℓ-ring (of which the ℓ-rings $C(X)$, $\mathcal{R}L$ or their bounded parts are of specific interest) is *increasing* (resp. *decreasing*) if $\alpha_\eta \leq \alpha_\xi$ (resp. $\alpha_\xi \leq \alpha_\eta$) whenever $\eta, \xi \in D$ with $\eta \leq \xi$. For any set X, let $\mathcal{P}(X)$, resp. $\mathcal{P}_{\text{fin}}(X)$, denote the powerset of X, resp. the set of all finite subsets of X. In this chapter, any reference to 'countable' is to be understood as 'finite or countably infinite'.

The idea of this section is to provide point-free counterparts to several theorems from Kundu and Raha [53]. We will begin by defining a suitable point-free counterpart for 'pointwise convergence' of a net of real continuous functions on a frame L to a continuous real function on L, at least in the case of monotone (i.e. increasing or decreasing) nets of functions, in terms of the cozero part of L.

Our motivation stems from the following observation: Let X be a topological space, $(f_\eta)_{\eta\in D}$ an increasing net in $C(X)$ and $f \in C(X)$ with $f_\eta \leq f$ for all $\eta \in D$.

Then

$$\begin{aligned} & (f_\eta)_{\eta\in D} \to f \text{ pointwisely} \\ \iff & \forall m \in \mathbb{N} : \bigcup_{\eta_0\in D} \bigcap_{\eta\in D, \eta\geq\eta_0} \{x \in X \mid |f(x) - f_\eta(x)| < \tfrac{1}{m}\} = X \\ \iff & \forall m \in \mathbb{N} : \bigcup_{\eta_0\in D} \{x \in X \mid f(x) - f_{\eta_0}(x) < \tfrac{1}{m}\} = X \\ \iff & \forall m \in \mathbb{N} : \bigcup_{\eta_0\in D} \{x \in X \mid (1 - m(f(x) - f_{\eta_0}(x))) > 0\} = X \\ \iff & \forall m \in \mathbb{N} : \bigcup_{\eta_0\in D} \{x \in X \mid (1 - m(f(x) - f_{\eta_0}(x))) \vee 0 \neq 0\} = X \end{aligned}$$

This then inspires the following definition:

Definition 6.1.1. For a frame L, $\alpha \in \mathcal{R}L$ and $(\alpha_\eta)_{\eta\in D}$ a net in $\mathcal{R}L$, we say that $(\alpha_\eta)_{\eta\in D}$ *increases everywhere to* α if $(\alpha_\eta)_{\eta\in D}$ is increasing, $\alpha_\eta \leq \alpha$ for all $\eta \in D$, and

$$\forall m \in \mathbb{N} : \bigvee_{\eta\in D} \operatorname{coz}((\mathbf{1} - m(\alpha - \alpha_\eta))^+) = 1.$$

We write this as $(\alpha_\eta)_{\eta\in D} \uparrow \alpha$.

Remark 6.1.2. Ball et al. [6] introduced a notion of 'pointwise supremum', resp. 'pointwise infimum' in $\mathcal{R}L$. Although both their definition and ours here were introduced independently and for different purposes, we can show without great difficulty that

$$(\alpha_\eta)_{\eta\in D} \uparrow \alpha \text{ if and only if } {\bigwedge}^{\bullet}\{\alpha - \alpha_\eta \mid \eta \in D\} = 0,$$

in the sense of [6]. This result follows from applying some well-known calculation rules in $\mathcal{R}L$ (see Preliminaries):

$$\begin{aligned} \forall m \in \mathbb{N}, \forall \eta \in D : & \operatorname{coz}((\mathbf{1} - m(\alpha - \alpha_\eta))^+) \\ & = (\mathbf{1} - m(\alpha - \alpha_\eta)(0, -) \\ & = m(\alpha - \alpha_\eta))(-, 1) \\ & = (\alpha - \alpha_\eta)(-, \tfrac{1}{m}). \end{aligned}$$

We will be using this equality throughout the chapter without mentioning it, but prefer to keep Definition 6.1.1 in its current form. The reason for this is because it is written in terms of Coz$L$, which ties in well with Lindelöf-type properties w.r.t. Coz$L$. We will turn to this topic later on.

In the same vein, we have

Definition 6.1.3. Let L be a frame, $\alpha \in \mathcal{R}L$ and $(\alpha_\eta)_{\eta\in D}$ a net in $\mathcal{R}L$. Then we say that $(\alpha_\eta)_{\eta\in D}$ *decreases everywhere to* α (and we write $(\alpha_\eta)_{\eta\in D} \downarrow \alpha$) if $(\alpha_\eta)_{\eta\in D}$ is decreasing, $\alpha \leq \alpha_\eta$ for all $\eta \in D$ and

$$\forall m \in \mathbb{N} : \bigvee_{\eta\in D} \operatorname{coz}((\mathbf{1} - m(\alpha_\eta - \alpha))^+) = 1.$$

Again a similar remark can be made as in the increasing case, namely that $(\alpha_\eta)_{\eta\in D} \downarrow \alpha$ if and only $\bigwedge^\bullet\{\alpha_\eta - \alpha \mid \eta \in D\} = 0$, in the sense of [6].

We can now introduce the following

Definition 6.1.4. For a frame L, we say that L satisfies the *Dini property* or *(DP)* if for any $\alpha \in \mathcal{R}L$ and any sequence $(\alpha_n)_n$ in $\mathcal{R}L$ which increases everywhere to α, the sequence $(\alpha_n)_n$ converges to α in the uniform topology on $\mathcal{R}L$.

Remark 6.1.5. We note that the fact that L satisfies (DP) can equivalently be stated as for any $\alpha \in \mathcal{R}L$ and any sequence $(\alpha_n)_n$ in $\mathcal{R}L$ which decreases everywhere to α, the sequence $(\alpha_n)_n$ converges to α in the uniform topology on $\mathcal{R}L$.

The reader will need little convincing, since for all $n \in \mathbb{N}$, $\alpha \leq \alpha_n$ is equivalent to $-\alpha \geq -\alpha_n$ for all $n \in \mathbb{N}$. Furthermore, $(\alpha_n)_n$ being decreasing is equivalent to $(-\alpha_n)_n$ being increasing, and for all $m \in \mathbb{N}$,

$$\bigvee_{n\in\mathbb{N}} \operatorname{coz}((\mathbf{1} - m(\alpha_\eta - \alpha))^+) = \bigvee_{n\in\mathbb{N}} \operatorname{coz}((\mathbf{1} - m((-\alpha) - (-\alpha_\eta))^+).$$

As a first result we obtain that, like in the pointed case, (DP) is equivalent to pseudocompactness:

Theorem 6.1.6. *For a frame L the following assertions are equivalent:*

(1) *L satisfies* (DP).

(2) *L is pseudocompact.*

Proof. We start with proving the implication (1) $\Rightarrow$ (2), so assume that L satisfies (DP). Since $|\alpha| = \alpha^+ + \alpha^-$, we see that α is bounded if and only if both α^+ and α^- are bounded, so we can assume for the rest of the proof that $\alpha \geq 0$. For each $n \in \mathbb{N}$, define $\alpha_n := \alpha \wedge \mathbf{n}$. Then clearly $\alpha_n \leq \alpha$ for all $n \in \mathbb{N}$ and $(\alpha_n)_n$ is increasing. Fix $m \in \mathbb{N}$. Using that $\mathcal{R}L$ is an ℓ-ring, it immediately follows that, for all $n \in \mathbb{N}$,

$$\alpha - \alpha \wedge \mathbf{n} = \alpha + (-\alpha) \vee (-\mathbf{n}) = (\alpha - \alpha) \vee (\alpha - \mathbf{n}) = (\alpha - \mathbf{n})^+$$

and because

$$(\alpha-\mathbf{n})^+(p, \tfrac{1}{m}) = \bigvee\{(\alpha-\mathbf{n})(r,s)\wedge\mathbf{0}(t,u) \mid \langle r,s\rangle \vee \langle t,u\rangle \subseteq \langle p, \tfrac{1}{m}\rangle\} \geq (\alpha-\mathbf{n})(p, \tfrac{1}{m}),$$

for all $p \in \mathbb{Q}$ with $p < 0$, we obtain that

$$\begin{aligned}
\bigvee_{n\in\mathbb{N}} \operatorname{coz}((\mathbf{1} - m(\alpha - \alpha \wedge \mathbf{n}))^+) &= \bigvee_{n\in\mathbb{N}} (\alpha - \alpha \wedge \mathbf{n})(-, \tfrac{1}{m}) \\
&\geq \bigvee_{n\in\mathbb{N}} (\alpha - \mathbf{n})(-, \tfrac{1}{m}) \\
&= \bigvee_{n\in\mathbb{N}} \alpha(-, n + \tfrac{1}{m}) \\
&= \alpha(\bigvee_{n\in\mathbb{N}} (-, n + \tfrac{1}{m})) \\
&= \alpha(1) = 1,
\end{aligned}$$

which proves that $(\alpha_n)_n \uparrow \alpha$. Because L satisfies (DP), $(\alpha_n)_n$ converges to α in the uniform topology, so there exists $n \in \mathbb{N}$ such that $\alpha - \alpha_n = |\alpha - \alpha \wedge \mathbf{n}| \leq \mathbf{1}$ and hence $\alpha \leq \mathbf{n} + \mathbf{1}$ finishing this part of the proof. In order to prove (2) $\Rightarrow$ (1), assume that L is pseudocompact and consider $(\alpha_n)_n \uparrow \alpha$ in $\mathcal{R}L$ and fix $m \in \mathbb{N}$. Then

$$\bigvee_{n\in\mathbb{N}} \operatorname{coz}((\mathbf{1} - m(\alpha - \alpha_n))^+) = 1,$$

and by recalling that a frame L is pseudocompact if and only if Coz$L$ is compact (see Prop. 2 from [10]), together with the fact that $(\alpha_n)_n$ is increasing (and therefore the cozero elements in the cover above grow larger with increasing n), yields that there exists an n_m in $\mathbb{N}$ with

$$\operatorname{coz}((\mathbf{1} - m(\alpha - \alpha_{n_m}))^+) = 1.$$

But since

$$\operatorname{coz}((\mathbf{1} - m(\alpha - \alpha_{n_m}))^+) = (\alpha - \alpha_{n_m})(-, \tfrac{1}{m}),$$

it then follows that

$$|\alpha - \alpha_{n_m}| = \alpha - \alpha_{n_m} \leq \frac{1}{\mathbf{m}}.$$

Now using that $(\alpha_n)_n$ is increasing and $m \in \mathbb{N}$ was chosen arbitrary, this proves that $(\alpha_n)_n$ converges to α in the uniform topology on $\mathcal{R}L$.

□

Definition 6.1.7. For a frame L, we say that L satisfies the *strong Dini Property* or *(sDP)* if for any $\alpha \in \mathcal{R}L$ and any net $(\alpha_\eta)_{\eta \in D}$ in $\mathcal{R}L$ which increases everywhere to α, the net $(\alpha_\eta)_{\eta \in D}$ converges to α in the uniform topology on $\mathcal{R}L$.

Remark 6.1.8. For the same reasons as mentioned previously, (sDP) can equivalently be defined by the statement that for any $\alpha \in \mathcal{R}L$ and any net $(\alpha_\eta)_{\eta \in D}$ in $\mathcal{R}L$ which decreases everywhere to α, the net $(\alpha_\eta)_{\eta \in D}$ converges to α in the uniform topology on $\mathcal{R}L$.

The following compactness-type properties for frames, which we will encounter throughout this chapter, are defined as obvious conservative extensions of their spatial counterparts (meaning that for a topological space X, the frame $\Omega(X)$ has the frame property if and only if X has the spatial property baring the same name):

Definition 6.1.9. A frame L is called

(a) *quasi-compact* (resp. *quasi-Lindelöf*) if every cover of L consisting of co-zero elements admits a finite (resp. countable) subcover.

(b) *initially-κ-compact* (resp. *initially-κ-quasi-compact*) if every cover of L of cardinality at most κ (resp. every cover of L consisting of cozero elements and of cardinality at most κ) admits a finite subcover, where κ is an infinite cardinal number.

(c) *initially-κ-Lindelöf* (resp. *initially-κ-quasi-Lindelöf*), if every cover of L of cardinality at most κ (resp. every cover of L consisting of cozero elements and of cardinality at most κ) admits a countable subcover, where κ is an infinite cardinal number.

Note that initially-$\aleph_0$-compact means *countably compact*. Intially-$\aleph_0$-quasi-compact is called *countably quasi-compact*.

Theorem 6.1.10. (AC) *For a frame L the following assertions are equivalent:*

(1) *L satisfies* (sDP).

(2) *L is quasi-compact.*

(3) *The coreflection of L from completely regular frames is compact.*

Proof. The equivalence of (2) and (3) is clear taking into account the following facts:

- a frame is completely regular if and only if it is $\bigvee$-generated by its cozero part,
- the coreflection of L from completely regular frames can be described as the subframe of L which is $\bigvee$-generated by $\mathrm{Coz}L$,
- a frame L and its coreflection from completely regular frames have the same cozero part.

We now turn to the implication (2) $\Rightarrow$ (1), so assume L satisfies (2) and let $(\alpha_\eta)_{n\in D}$ be an increasing net in $\mathcal{R}L$ and $\alpha \in \mathcal{R}L$ with $(\alpha_\eta)_{n\in D} \uparrow \alpha$. The proof now continues along the same lines as the proof of the implication (2) $\Rightarrow$ (1) in 6.1.6, with the only difference that we now need to use the directedness of the indexing set D in order to produce, given a finite cover

$$\mathrm{coz}((\mathbf{1} - m(\alpha - \alpha_{\eta_1}))^+) \vee \cdots \vee \mathrm{coz}((\mathbf{1} - m(\alpha - \alpha_{\eta_n}))^+) = 1$$

with $\eta_1, \ldots, \eta_n \in D$, an index $\eta_0 \in D$ with $\eta_1, \ldots, \eta_n \leq \eta_0$, for which then automatically $\mathrm{coz}((\mathbf{1} - m(\alpha - \alpha_{\eta_0}))^+) = 1$ holds, in order to conclude that $(\alpha_\eta)_{\eta\in D}$

converges to α uniformly.

Finally, to prove (1) $\Rightarrow$ (2), assume that L satisfies (sDP) and take $F \subseteq \mathrm{Coz}L$ with $\bigvee F = 1$. For each $a \in F$, pick $\alpha_a \in \mathcal{R}L$ such that $0 \leq \alpha_a \leq \mathbf{1}$ and $a = \mathrm{coz}(\alpha_a)$ (note that we can always assure that $0 \leq \alpha_a \leq \mathbf{1}$, since $\mathrm{coz}(\beta) = \mathrm{coz}(\beta^2 \wedge \mathbf{1})$ for all $\beta \in \mathcal{R}L$). By a well-known fact in $\mathcal{R}L$ which can be found in [8], we have for all $a \in F$ that

$$\mathrm{coz}(\alpha_a) = \bigvee_n \mathrm{coz}((n\alpha_a - \mathbf{1})^+) = \bigvee_n \mathrm{coz}((n\alpha_a - \mathbf{1})^+ \wedge \mathbf{1}).$$

Now put

$$D := \mathcal{P}_{\mathrm{fin}}(F \times \mathbb{N}),$$

order D by subset inclusion and for every $\eta \in D$, define

$$\beta_\eta := \bigvee \{(n\alpha_a - \mathbf{1})^+ \wedge \mathbf{1} \mid (a, n) \in \eta\},$$

where the join is in fact finite and therefore exists in $\mathcal{R}L$. Since for all $m \in \mathbb{N}$,

$$\begin{aligned}
&\bigvee_{\eta \in D} \mathrm{coz}((\mathbf{1} - m(\mathbf{1} - \beta_\eta))^+) \\
&= \bigvee_{\eta \in D} \mathrm{coz}((\mathbf{1} - m(\mathbf{1} - \bigvee_{(a,n)\in\eta} (n\alpha_a - \mathbf{1})^+ \wedge \mathbf{1}))^+) \\
&= \bigvee_{\eta \in D} (\mathbf{1} - \bigvee_{(a,n)\in\eta} (n\alpha_a - \mathbf{1})^+ \wedge \mathbf{1})(-, \tfrac{1}{m}) \\
&= \bigvee_{\eta \in D} (\bigvee_{(a,n)\in\eta} (n\alpha_a - \mathbf{1})^+ \wedge \mathbf{1})(1 - \tfrac{1}{m}, -) \\
&\geq \bigvee_{\eta \in D} \bigvee_{(a,n)\in\eta} ((n\alpha_a - \mathbf{1})^+ \wedge \mathbf{1})(1 - \tfrac{1}{m}, -) \\
&= \bigvee_{\eta \in D} \bigvee_{(a,n)\in\eta} (n\alpha_a - \mathbf{1})^+(1 - \tfrac{1}{m}, -) \\
&= \bigvee_{\eta \in D} \bigvee_{(a,n)\in\eta} (n\alpha_a - \mathbf{1})(1 - \tfrac{1}{m}, -)
\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{\eta \in D} \bigvee_{(a,n)\in\eta} \alpha_a(\tfrac{2}{n} - \tfrac{1}{mn}, -) \\
&\geq \bigvee_{a \in F} \bigvee_{n\in\mathbb{N}} \alpha_a(\tfrac{2}{n} - \tfrac{1}{mn}, -) \\
&= \bigvee_{a\in F} \operatorname{coz}(\alpha_a) \\
&= 1,
\end{aligned}
$$

we have that $(\beta_\eta)_{\eta\in D} \uparrow \mathbf{1}$ and hence, by (sDP), that $(\beta_\eta)_{\eta\in D}$ converges to $\mathbf{1}$ in the uniform topology. Therefore, there exists $\eta \in D$, $\eta \neq \emptyset$, such that $\mathbf{1} - \beta_\eta \leq \frac{1}{2}$, or equivalently, $\beta_\eta \geq \frac{1}{2}$. Consequently,

$$\bigvee_{(a,n)\in\eta} (n\alpha_a - \mathbf{1})^+ \geq \frac{1}{2}.$$

In a general lattice-ordered ring R, we have that

$$\forall r, s \in R^+ : r \vee s \leq r \wedge s + r \vee s = r + s,$$

so we can deduce from the last inequality that

$$\sum_{(a,n)\in\eta} (n\alpha_a - \mathbf{1})^+ \geq \frac{1}{2}$$

and hence

$$\operatorname{coz}\Big(\sum_{(a,n)\in\eta} (n\alpha_a - \mathbf{1})^+\Big) = 1.$$

Since

$$\operatorname{coz}\Big(\sum_{(a,n)\in\eta} (n\alpha_a - \mathbf{1})^+\Big) = \bigvee_{(a,n)\in\eta} \operatorname{coz}((n\alpha_a - \mathbf{1})^+) \leq \bigvee_{(a,n)\in\eta} \operatorname{coz}(\alpha_a) = \bigvee_{(a,n)\in\eta} a,$$

this completes the proof.

□

The strengthening of (DP) to (sDP) was brought about by generalising the sequence $(\alpha_n)_n$ in $\mathcal{R}L$ to a net $(\alpha_\eta)_{\eta\in D}$. Since a frame L is pseudocompact if and only if Coz$L$ is compact (see [10]), one notices that, with reference to compactness, this results in countable joins (of cozero elements) being strengthened to arbitrary joins (of cozero elements).

Next we consider different (infinite) cardinalities of D for the net $(\alpha_n)_n$ in $\mathcal{R}L$. First we start out with a definition:

Definition 6.1.11. For a frame L and an infinite cardinal number κ, we say that L satisfies the *κ-Dini property* or *(κ-DP)* if for any $\alpha \in \mathcal{R}L$ and any net $(\alpha_\eta)_{\eta\in D}$ in $\mathcal{R}L$ with cardinality of D at most κ and which increases everywhere to α, the net $(\alpha_\eta)_{\eta\in D}$ converges to α in the uniform topology on $\mathcal{R}L$.

Remark 6.1.12. Note that as in 6.1.5 the fact that L satisfies (κ-DP) is equivalent to the statement that for any $\alpha \in \mathcal{R}L$ and any net $(\alpha_\eta)_{\eta\in D}$ in $\mathcal{R}L$ with cardinality of D at most κ and which decreases everywhere to α, the net $(\alpha_\eta)_{\eta\in D}$ converges to α in the uniform topology on $\mathcal{R}L$.

We now immediately obtain a characterization of (κ-DP):

Theorem 6.1.13. (AκC) *For a frame L and an infinite cardinal number κ, the following assertions are equivalent:*

(1) *L satisfies* (κ-DP).

(2) *L is initially-κ-quasi-compact.*

Proof. One only has to note that the proof of 6.1.10 can be redone since for κ infinite and $\mathrm{Card}(F) \leq \kappa$,

$$\mathrm{Card}(\mathcal{P}_{\mathrm{fin}}(F \times \mathbb{N})) = \mathrm{Card}(\bigcup_{n\in\mathbb{N}} (F \times \mathbb{N})^n) \leq \kappa.$$

□

Remark 6.1.14. Note that in this case a counterpart to 6.1.10(3) is not available.

Corollary 6.1.15. *For a frame L the following assertions are equivalent:*

(1) *L satisfies* (DP).

(2) *L satisfies* ($\aleph_0$-DP).

In view of the previous, we now have proved the following results:

Theorem 6.1.16. (AC) *For a frame L, the following assertions are equivalent:*

(1) *L satisfies* (sDP).

(2) *L is quasi-Lindelöf and L satisfies* (DP).

Proof. Obvious. $\square$

Theorem 6.1.17. (AκC) *For a frame L and an infinite cardinal number κ, the following assertions are equivalent:*

(1) *L satisfies* (κ-DP).

(2) *L is initially-κ-quasi-Lindelöf and L satisfies* (DP).

Proof. Obvious. $\square$

Next we give a point-free counterpart of the result by Kundu and Raha [53] which characterised quasi-Lindelöfness of a space in terms of continuous real functions.

Theorem 6.1.18. (AC) *For a frame L, the following assertions are equivalent:*

(1) *L is quasi-Lindelöf.*

(2) *The coreflection of L from completely regular frames is Lindelöf.*

(3) *For any net $(\alpha_\eta)_{\eta\in D}$ in $\mathcal{R}L$ and any $\alpha \in \mathcal{R}L$ such that $(\alpha_\eta)_{\eta\in D} \uparrow \alpha$, there exists an increasing sequence $(\eta_n)_n$ in D such that $(\alpha_{\eta_n})_n \uparrow \alpha$.*

Proof. The equivalence of (1) and (2) is clear. In order to prove (1) $\Rightarrow$ (3), assume L is quasi-Lindelöf and take a net $(\alpha_\eta)_{\eta\in D}$ in $\mathcal{R}L$ and $\alpha \in \mathcal{R}L$ such that $(\alpha_\eta)_{\eta\in D} \uparrow \alpha$. By definition, we have for all $m \in \mathbb{N}$ that

$$\bigvee_{\eta\in D} \operatorname{coz}((\mathbf{1} - m(\alpha - \alpha_\eta))^+) = 1,$$

so the quasi-Lindelöfness of L implies that for each $m \in \mathbb{N}$, there exist $D_m \subseteq D$ with D_m countable such that

$$\bigvee_{\eta \in D_m} \operatorname{coz}((\mathbf{1} - m(\alpha - \alpha_\eta))^+) = 1.$$

Then

$$D' := \bigcup_{m \in \mathbb{N}} D_m$$

is also countable. Therefore there exists a sequence $(\eta_k)_k$ in D such that $D' = \{\eta_k \mid k \in \mathbb{N}\}$ (if D is countably infinite, we can take the sequence $(\eta_k)_k$ to be injective and otherwise, we take it to be injective till D is exhausted and then make it constant at the last occurring value). Now put $\tilde{\eta}_1 := \eta_1$ and, by directedness of D, inductively choose $\tilde{\eta}_{k+1} \in D$ with $\eta_1, \ldots, \eta_k, \tilde{\eta}_k \leq \tilde{\eta}_{k+1}$. Then $(\alpha_{\tilde{\eta}_k})_k$ is an increasing sequence in $\mathcal{R}L$. We now show that

$$(\alpha_{\tilde{\eta}_k})_k \uparrow \alpha.$$

Fix $m \in \mathbb{N}$, then

$$\bigvee_{k \in \mathbb{N}} \operatorname{coz}((\mathbf{1} - m(\alpha - \alpha_{\eta_k}))^+) \geq \bigvee_{\eta \in D_m} \operatorname{coz}((\mathbf{1} - m(\alpha - \alpha_\eta))^+) = 1.$$

But for each $k \in \mathbb{N}$, we have $\eta_k \leq \tilde{\eta}_k$ and hence $\alpha_{\eta_k} \leq \alpha_{\tilde{\eta}_k}$, which yields that for all $k \in \mathbb{N}$,

$$\operatorname{coz}((\mathbf{1} - m(\alpha - \alpha_{\eta_k}))^+) \leq \operatorname{coz}((\mathbf{1} - m(\alpha - \alpha_{\tilde{\eta}_k}))^+)$$

and therefore

$$\bigvee_{k \in \mathbb{N}} \operatorname{coz}((\mathbf{1} - m(\alpha - \alpha_{\tilde{\eta}_k}))^+) = 1.$$

To complete the proof, we show that (3) $\Rightarrow$ (1), so assume (3) holds and take $F \subseteq \operatorname{Coz}L$ with $\bigvee F = 1$. For each $a \in F$, fix $\alpha_a \in \mathcal{R}L$ with $0 \leq \alpha_a \leq \mathbf{1}$ and $a = \operatorname{coz}(\alpha_a)$. Like in the proof of 6.1.10, define

$$D := \mathcal{P}_{\text{fin}}(F \times \mathbb{N}),$$

order D by subset inclusion and for every $\eta \in D$, define

$$\beta_\eta := \bigvee \{(n\alpha_a - \mathbf{1})^+ \wedge \mathbf{1} \mid (a, n) \in \eta\},$$

where the join is in fact finite and therefore exists in $\mathcal{R}L$. Then it follows from the proof of Theorem 6.1.10 that $(\beta_\eta)_{\eta\in D} \uparrow \mathbf{1}$, and applying (3) yields the existence of an increasing sequence $(\eta_k)_k$ in D such that

$$(\beta_{\eta_k})_k \uparrow \mathbf{1}.$$

This means that for every $m \in \mathbb{N}$,

$$\bigvee_{k\in\mathbb{N}} \operatorname{coz}((\mathbf{1} - m(\mathbf{1} - \beta_{\eta_k}))^+) = 1$$

or, more explicitly,

$$\bigvee_{k\in\mathbb{N}} \operatorname{coz}((\mathbf{1} - m(\mathbf{1} - \bigvee\{(n\alpha_a - \mathbf{1})^+ \wedge \mathbf{1} \mid (a,n) \in \eta_k\}))^+) = 1.$$

In particular, we get for $m = 1$ that

$$\bigvee_{k\in\mathbb{N}} \operatorname{coz}(\bigvee\{(n\alpha_a - \mathbf{1})^+ \wedge \mathbf{1} \mid (a,n) \in \eta_k\}) = 1.$$

Applying the same ℓ-ring trick as in the proof of 6.1.10 yields that

$$\bigvee_{k\in\mathbb{N}} \operatorname{coz}(\sum_{(a,n)\in\eta_k} ((n\alpha_a - \mathbf{1})^+ \wedge \mathbf{1})) = 1,$$

from which we infer that

$$\bigvee_{k\in\mathbb{N}} \bigvee_{(a,n)\in\eta_k} \operatorname{coz}((n\alpha_a - \mathbf{1})^+ \wedge \mathbf{1}) = 1.$$

A second use of the formula

$$\operatorname{coz}(\alpha_a) = \bigvee_{n'} \operatorname{coz}((n'\alpha_a - \mathbf{1})^+) = \bigvee_{n'} \operatorname{coz}((n'\alpha_a - \mathbf{1})^+ \wedge \mathbf{1})$$

from [8] yields

$$\bigvee_{k\in\mathbb{N}} \bigvee_{(a,n)\in\eta_k} a = \bigvee_{k\in\mathbb{N}} \bigvee_{(a,n)\in\eta_k} \operatorname{coz}(\alpha_a) = 1.$$

So

$$S := \{a \in F \mid (a,n) \in \eta_k, \text{ for some } n, k \in \mathbb{N}\}$$

is a countable (since all η_k are finite) subcover of F, showing the quasi-Lindelöfness of L.

□

It would seem that as a result of generalising the finite subcovers (as with quasi-compactness) to countable ones (as with quasi-Lindelöfness), the nets $(\alpha_\eta)_{\eta\in D}$ lose their uniform convergence. Next we show that the equivalence of the assertions in Theorem 6.1.18 is preserved under different (infinite) cardinalities for the nets/covers:

Theorem 6.1.19. (AκC) *For a frame L and an infinite cardinal number κ, the following assertions are equivalent:*

(1) *L is initially κ-quasi-Lindelöf.*

(2) *For any net $(\alpha_\eta)_{\eta\in D}$ with cardinality of D at most κ in $\mathcal{R}L$ and any $\alpha \in \mathcal{R}L$ such that $(\alpha_\eta)_{\eta\in D} \uparrow \alpha$, there exists an increasing sequence $(\eta_n)_n$ in D such that $(\alpha_{\eta_n})_n \uparrow \alpha$.*

Proof. One only has to note that the proof of 6.1.18 can be redone since for κ infinite and $\mathrm{Card}(F) \leq \kappa$,

$$\mathrm{Card}(\mathcal{P}_{\mathrm{fin}}(F \times \mathbb{N})) = \mathrm{Card}(\bigcup_{n\in\mathbb{N}} (F \times \mathbb{N})^n) \leq \kappa.$$

□

Remark 6.1.20. (i) Note that in this case a counterpart to 6.1.18(2) is not available.

(ii) For Theorems 6.1.18 and 6.1.19 we could equally well have replaced every occurence of "$(\alpha_\eta)_{\eta\in D} \uparrow \alpha$" with "$(\alpha_\eta)_{\eta\in D} \downarrow \alpha$" and the assertions would have remained equivalent. The proof of the descending case is analogous to the ascending case.

Definition 6.1.21. A frame L is called *quasi-almost-compact* if for every cover F of L consisting of cozero elements, there exists $S \subseteq F$ finite such that

$$(\bigvee S)^* = 0.$$

If κ is an infinite cardinal number, a frame L is called *initially-κ-almost-compact* (resp. *initially-κ-quasi-almost-compact*) if for every cover F of L of cardinality at most κ (resp. for every cover F of L consisting of cozero elements and of cardinality at most κ), there exists $S \subseteq F$ finite such that

$$(\bigvee S)^* = 0.$$

Initially-$\aleph_0$-almost-compact (resp. initially-$\aleph_0$-quasi-almost-compact) is *countably almost-compact* (resp. *countably quasi-almost-compact*).

Proposition 6.1.22. *Every quasi-almost-compact frame satisfies* (sDP). *For any infinite cardinal number κ, every initially-κ-quasi-almost-compact frame satisfies* (κ-DP).

Proof. Take L a quasi-almost-compact frame. First note that in order prove (sDP), it suffices to show (as the two assertions are equivalent) that nets in $\mathcal{R}L$ descending everywhere to $\mathbf{0}$, converge to $\mathbf{0}$ in the uniform topology. So assume a net $(\alpha_\eta)_{\eta\in D}$ such that $(\alpha_\eta)_{\eta\in D} \downarrow \mathbf{0}$. Fix $m \in \mathbb{N}$. Then

$$\bigvee_{\eta\in D} \operatorname{coz}((\mathbf{1} - m\alpha_\eta)^+) = 1,$$

so applying quasi-almost compactness of L and the fact that $(\alpha_\eta)_{\eta\in D}$ is descending yields that there exists $\eta_0 \in D$ with

$$(\operatorname{coz}((\mathbf{1} - m\alpha_{\eta_0})^+))^* = 0.$$

This means that, with

$$a := \operatorname{coz}((\mathbf{1} - m\alpha_{\eta_0})^+),$$

the frame homomorphism

$$\hat{a} = (\cdot) \wedge a : L \to \downarrow a$$

is dense. Because

$$(m\alpha_{\eta_0} - \mathbf{1})(0, -) = \alpha_{\eta_0}(\tfrac{1}{m}, -)$$

and

$$a = ((\mathbf{1} - m\alpha_{\eta_0})^+)(0, -) = (\mathbf{1} - m\alpha_{\eta_0})(0, -) = \alpha_{\eta_0}(-, \tfrac{1}{m})$$

it follows that

$$\hat{a} \circ (m\alpha_{\eta_0} - \mathbf{1})(0, -) = \alpha_{\eta_0}(\tfrac{1}{m}, -) \wedge \alpha_{\eta_0}(-, \tfrac{1}{m}) = \alpha_{\eta_0}((\tfrac{1}{m}, -) \wedge (-, \tfrac{1}{m})) = \alpha_{\eta_0}(0).$$

Since $\alpha_{\eta_0}(0) = 0$, we have that $\hat{a} \circ (m\alpha_{\eta_0} - \mathbf{1}) \leq 0$. Therefore $\hat{a} \circ (m\alpha_{\eta_0} - \mathbf{1})^+ = 0$ and because $\hat{a}$ is dense, $(m\alpha_{\eta_0} - \mathbf{1})^+ = 0$, which yields that

$$0 \leq \alpha_{\eta_0} \leq \frac{1}{\mathbf{m}}.$$

Because m was taken arbitrary and $(\alpha_\eta)_{\eta \in D}$ is descending this completes the proof for quasi-almost compact L. The proof for initially κ-quasi-almost-compact L is the same.

□

Summarizing what we have proved up till now, we obtain the following theorems concerning (sDP) and (κ-DP):

Theorem 6.1.23. (AC) *For a frame L the following assertions are equivalent:*

(1) *L satisfies* (sDP)*.*

(2) *L is quasi-Lindelöf and satisfies* (DP)*.*

(3) *L is quasi-compact.*

(4) *The coreflection of L from completely regular frames is compact.*

(5) *L is quasi-almost-compact.*

(6) *The coreflection of L from completely regular frames is almost compact.*

Theorem 6.1.24. (AκC) *For a frame L and an infinite cardinal number κ, the following assertions are equivalent:*

(1) *L satisfies* (κ-DP)*.*

(2) *L is initially-κ-quasi-Lindelöf and satisfies* (DP)*.*

(3) *L is initially-κ-quasi-compact.*

(4) *L is initially-κ-quasi-almost-compact.*

Specializing 6.1.23 to completely regular frames, one obtains:

Corollary 6.1.25. (AC) *For a completely regular frame L the following assertions are equivalent:*

(1) *L satisfies* (sDP).

(2) *L is compact.*

(3) *L is almost compact.*

Note that the equivalence of (2) and (3), already true for regular L, is well-known from Hong [46, 47] and Paseka and Šmarda [57].

Regarding (DP), putting together what we now know yields:

Theorem 6.1.26. *For a frame L, the following assertions are equivalent:*

(1) *L satisfies* (DP).

(2) *L satisfies* ($\aleph_0$-DP).

(3) *L is countably quasi-compact.*

(4) *L is countably quasi-almost-compact.*

(5) *L is pseudocompact.*

The equivalence of (3) and (5) is a result of Banaschewski and Gilmour [10] and is in fact used in our proof of 6.1.6. We also recall from Banaschewski et el. [11] that (assuming (AC)) for completely regular L, pseudocompactness of L is equivalent to countable almost compactness.

6.2 On the Stone-Weierstrass property for completely regular frames

Throughout this section L will always be assumed to be completely regular. We recall from [8] that with $\mathbb{R}$ denoting the Dedekind real numbers, we have that $\mathbb{R} \simeq \Sigma(\mathcal{L}(\mathbb{R}))$ (as topological spaces) and that also $\mathbb{R} \simeq \mathcal{R}\mathbf{2}$ (as lattice-ordered

rings). If L is an arbitrary frame, the elements of $\mathcal{R}\mathbf{2}$, which can be identified with those $\alpha \in \mathcal{R}(L)$ that factor through the unique frame homomorphism $\mathbf{2} \to L$, form a subring of $\mathcal{R}L$ (in fact of $\mathcal{R}^*L$) that is isomorphic to $\mathbb{R}$, and therefore $\mathcal{R}L$ (and also $\mathcal{R}^*L$) can be viewed as $\mathbb{R}$-algebras.
In Banaschewski [8, 9] the following definition of separating subalgebra (of $\mathcal{R}^*L$) was given as a point-free counterpart to the classical concept of point-separating subalgebra (of $C^*(X)$): For L a completely regular frame, an $\mathbb{R}$-subalgebra A of $\mathcal{R}^*L$ is called *separating* if

$$\mathrm{coz}[A] = \{\mathrm{coz}(\alpha) \mid \alpha \in A\}$$

$\bigvee$-generates L.
Analogous to the classical pointed definition we now introduce the following point-free version of the Stone-Weierstrass Property:

Definition 6.2.1. A completely regular frame L is said to have the *Stone-Weierstrass property* or *(SWP)* if every separating unital $\mathbb{R}$-subalgebra of $\mathcal{R}^*L$ is dense in $\mathcal{R}^*L$ with respect to the uniform topology.

With the notion of separatedness quoted above, Banaschewski proved a point-free Stone-Weierstrass Theorem (see Banaschewski [8, 9])which in view of 6.2.1 reads as follows:

All compact completely regular frames satisfy (SWP).

We write $\mathbf{K}$ for the category of compact completely regular frames, and $\mathbf{A}$ for the category of unital archimedean f-rings. For completeness's sake, recall that for a completely regular frame L the the Stone-Čech compactification of L, i.e. the coreflection of L from $\mathbf{K}$, is given by

$$\beta_L : \mathcal{M}(\mathcal{R}^*L) \to L, J \mapsto \bigvee \mathrm{coz}[J]$$

where $\mathcal{M}(\mathcal{R}^*L)$ is the frame of all closed l-ideals of $\mathcal{R}^*L$ (see Banaschewski [8] for details about this construction and Banaschewski and Sioen [17] for an alternative construction based on Jacobson radical ideals). If one writes $\beta L = \mathcal{M}(\mathcal{R}^*L)$, then

the compactification $\beta_L : \beta L \to L$ is uniquely determined among all compactifications of L by the property that every $\alpha \in \mathcal{R}^*L$ has a unique extension to βL, i.e. there exists a unique $\overline{\alpha} : \mathcal{L}(\mathbb{R}) \to \beta L$ such that $\alpha = \beta_L\overline{\alpha}$.
Following Banaschewski and Sioen [18] we write, for a completely regular frame L, $\Delta(\mathbf{K} \downarrow L)$ for the category of all (completely regular) compactifications of L (we recall that a *completely regular compactification of* L is a dense onto frame homomorphism $h : M \to L$ with M compact completely regular); and we call an l-subring A of $\mathcal{R}^*(L)$ a *K-ring of* L if A is complete (or equivalently, closed) with respect to its natural uniformity (i.e. the uniform uniformity), contains the constant functions and is separating (i.e. $\{\mathrm{coz}(\alpha) \mid \alpha \in A\}$ $\bigvee$-generates L). The partially ordered set of K-rings of L, considered as a category is denoted as $\mathbf{KRg}(L)$.

We collect the following ingredients, before we can come to our final conclusion:

(a) From Banaschewski and Sioen [18]; For a completely regular frame L, the categories $\Delta(\mathbf{K} \downarrow L)$ and $\mathbf{KRg}(L)$ are equivalent. We refer to [18] for more details.

(b) From Banaschewski [7]: a (completely regular) frame L has a minimal compactification, i.e. with the terminology introduced here, the partial order associated with the preorder $\Delta(\mathbf{K} \downarrow L)$ has a bottom, if and only if L is a continuous frame. This is the point-free counterpart of the well-known result that a Tychonoff space admits a smallest compactification (which is the Alexandroff compactification) if and only if it is locally compact. We refer the reader to [7] for more details.

(c) From Banaschewski [9]: If A is a bounded f-ring over $\mathbb{Q}$ and S is a uniformly closed subring of A containing $\mathbb{Q}$, then S is a sublattice of A.

Combining these elements together, we now obtain the following characterisation of (SWP):

Theorem 6.2.2. *For a completely regular frame L, the following assertions are equivalent:*

(1) *L satisfies* (SWP)

(2) *$\mathcal{R}^*L$ is the only K-ring of L*

(3) *β_L is (up to isomorphisms fixing L) the only compactification of L.*

(4) *L is continuous and every $f \in \mathcal{R}^*L$ extends uniquely to the minimal compactification of L.*

Proof. It is well-known that $\mathcal{R}^*(L)$ is a K-ring of L, which corresponds to β_L. The equivalence of (1) and (2) is clear using (c), whereas the equivalence of (2) and (3) immediately follows from (a) because an equivalence of categories preserves isomorphisms, and isomorphisms in $\mathbf{KRg}(L)$ are equalities. That (3) and (4) are equivalent is a direct consequence of (b). □

Remark 6.2.3. (i) The frame of opens of the deleted Tychonoff plank provides a (spatial) example of a completely regular frame satisfying (4) (and hence all the equivalent properties in 6.2.2), since it is shown in Steen and Seebach, Jr. [69] that every real-valued continuous function on the deleted Tychonoff plank extends (uniquely) to a real-valued continuous function on the Tychonoff plank and since the deleted Tychonoff plank is locally compact, being a dense open subspace of the compact Hausdorff Tychonoff plank. If one assumes (AC), all continuous completely regular locales are spatial, so assuming (AC) we see here that the point-free (SWP) implies spatiality.

(ii) Comparing 6.2.2 with the results from the last paragraph in Kundu and Raha [53], one immediately observes that the point-free Stone-Weierstrass Property (SWP) defined in 6.2.1, when applied to the open set lattice $\Omega(X)$ of a Tychonoff space is not equivalent to the classical, spatial, Stone-Weierstrass property alluded to in Question 2 (see Holgate et al. [44]) and treated e.g. in Hewitt [40], Stephenson, Jr. [70] and, Kundu and Raha [53]. This has to do with the fact that separatedness of an $\mathbb{R}$-subalgebra of $\mathcal{R}^*\Omega(X)$ is in general *stronger* than the assertion that the corresponding $\mathbb{R}$-subalgebra of $C^*(X)$ separates points; for compact Hausdorff X it was shown in Banaschewski [8,9] that the two notions coincide for closed $\mathbb{R}$-subalgebras. So for a Tychonoff space X, the assertion that $\Omega(X)$ satisfies the point-free Stone-Weierstrass Property (SWP) from 6.2.1 is *weaker* than the assertion that X satisfies the classical Stone-Weierstrass property discussed in Question 2, and the latter is known to be equivalent to compactness of X from Hewitt [40].

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